A Study of Isotropic Turbulence

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Abstract

The prime difficulty in finding a solution to the problem of high Reynolds' number turbulence is the large number of strongly interacting degrees of freedom. The most natural method of tackling the problem is to perform a Renormalisation Group transformation which removes from consideration the small scale degrees of freedom. The historical development of this approach is charted and the importance of the universal nature of the inertial range behaviour is noted.

The problem tackled initially is that of homogeneous, isotropic, stationary turbulence. The velocity field is split into explicit(large) and implicit(small) scale modes and then the implicit modes are split into two parts, which correspond to the correlated and uncorrelated behaviour of the implicit modes with regard to the explicit modes. A new averaging procedure is introduced where the realisations of the ensemble are those where the explicit modes are held constant while the implicit ones explore the possible solutions of the Navier-Stokes equation(NSE). The equation for the modes averaged away yields a term which acts as an increment to the viscosity in the equation for the explicit modes. The form of this term is arrived at by considering the uncorrelated part of the implicit field as the leading contribution. It is argued that the time scales of the explicit scales will be much greater than those of the implicit scales facilitating a Markovian approximation. Further, the sum of the moment hierarchy in the uncorrelated part can be shown to be approximated by its first term within a plausible model. Hence, a form for the increment to viscosity is obtained. The renormalisation group iteration yields a fixed point in the effective viscosity. It is argued that this fixed point will have captured the essential physics of the inertial range.

In order to make calculations, an ansatz is made for the uncorrelated part of the implicit field. By taking a truncated Taylor series in the implicit scales about the dissipation cut-off, it is found that the ansatz approximates the uncorrelated field.

The fixed point for the effective viscosity is calculated numerically. From this, a value for the Kolmogorov constant of 1.60 is calculated. This constant shows the expected behaviour for variation of the width of the band of wavenumbers averaged out at each iteration. There is a region of parameter space where the constant shows no dependence on the input parameters. This region is taken to produce valid results, since a real quantity cannot depend on how the average is performed.

The method is extended to the problem of transport of a passive scalar field by a turbulent velocity field. A procedure analogous to that above which splits both the scalar and velocity fields is used. The results are similar to those for the velocity field in that the calculated universal constant, the Obukhov-Corrsin constant, is found not to depend on non-physical parameters in a certain range. This constant is found to have a value 1.02, in good agreement with the best available simulation results.

Contents

Chapter 1

Introduction

1.1 Turbulence

Turbulence is a complex phenomenon seen to occur under certain circumstances in the behaviour of fluids. It is characterised by the appearance of a wide range of active Fourier modes in the velocity field of the fluid (i.e. dynamically significant behaviour goes on at many different length scales). Taking a time series measurement of velocity in such a flow would yield a highly irregular signal without obvious structure.

The phenomenon was first identified over one hundred years ago by Reynolds when he noted that under certain circumstances a dye tracer in fluid no longer followed a simple line characteristic of laminar flow but instead mixed rapidly with the fluid. This type of behaviour is commonly seen if one watches smoke rising from a cigar; the initial column-like rise develops waves then swirls and these swirls (or eddies) breakdown into smaller eddies which interweave and create yet more complex patterns, finally leaving one struggling to make out any structures at all. However, though turbulence may be chaotic, it is not beyond description as

certain processes can be singled as fundamental to its development.

The governing equation for the behaviour of a fluid is the Navier-Stokes equation(NSE):

$$\left[\frac{\partial}{\partial t} + \underline{u}.\underline{\nabla}\right] \underline{u}(\mathbf{x},t) = -\frac{1}{\rho}\underline{\nabla} p(\mathbf{x},t) + \nu_0 \nabla^2 \underline{u}(\mathbf{x},t), \tag{1.1}$$

where p is the pressure in the fluid, ρ is the fluid density and ν_0 is the kinematic viscosity of the fluid(diffusivity of momentum). This work will only be concerned with incompressible fluids (i.e. those whose density is constant), implying by the continuity equation that

$$\underline{\nabla}.\underline{u} = 0, \tag{1.2}$$

in other words the velocity field is solenoidal.

The form of the NSE used assumes a particular constitutive relationship between stress and rate of strain which is associated with fluids called Newtonian. These are the most commonly occurring and will be the type dealt with herein.

I have mentioned that turbulence occurs under certain circumstances and now will outline which variables govern the transition from laminar to turbulent flow. Reynolds discovered that transition occurs above a value for a certain dimensionless number, called the Reynolds number

$$Re = \frac{\overline{u}d}{\nu_0}. \tag{1.3}$$

Here, $\overline{u}$ and d are velocity and length scales representative of the dynamics of the flow (e.g. in pipe flow they would be the bulk mean velocity and the diameter of the pipe, respectively). This dimensionless ratio can be considered as a ratio of the inertial to viscous terms in the NSE:

$$Re \simeq \frac{|(\underline{u}.\underline{\nabla})\underline{u}|}{|\nu_0 \nabla^2 \underline{u}|}, \tag{1.4}$$

provided $\overline{u}$ and d are representative and the flow is steady. The dominance of one of these terms over the other dictates the type of behaviour the fluid exhibits. If

Re is small viscous forces dominate and there is no time for complex structure to develop before dissipation. If Re is large, however, the nonlinear inertial term dominates and the possibility of non-local effects appears.

It is the presence of the nonlinearity which gives the problem of turbulence its richness of structure as it provides a means for coupling dynamical behaviour on many length scales together. However, it is the presence of this term which prevents simple mathematical solutions to the high Reynolds' number situation.

In order to simplify the mathematics of the problem work will be done in Fourier wavenumber space. Using

$$u_\alpha(\mathbf{x},t) = \int d^3\, k u_\alpha(\mathbf{k},t) exp(ik_\beta x_\beta) \tag{1.5}$$

and a similar equation for the pressure, the NSE is transformed to

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha(\mathbf{k},t) = -ik_\beta \int d^3 j u_\alpha(\mathbf{j},t) u_\beta(\mathbf{k}-\mathbf{j},t) - ik_\alpha p(\mathbf{k},t) \tag{1.6}$$

in kinematic units (where $\rho = 1$). The Einstein summation convention is assumed, unless otherwise stated.

The pressure can be eliminated using the continuity equation, which transforms to

$$k_\alpha u_\alpha(\mathbf{k},t) = 0, \tag{1.7}$$

to give the normal form of the NSE in Fourier space

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3 j u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t) + f_\alpha(\mathbf{k},t), \tag{1.8}$$

where

$$M_{\alpha\beta\gamma}(\mathbf{k}) = (2i)^{-1} \left[k_\beta D_{\alpha\gamma}(\mathbf{k}) + k_\gamma D_{\alpha\beta}(\mathbf{k})\right], \tag{1.9}$$

and

$$D_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta |\mathbf{k}|^{-2} \tag{1.10}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. $f_\alpha(\mathbf{k},t)$ is the Fourier transform of the function which represents the action of stirring forces on the fluid.

Equation (1.8) is the divergenceless equation of motion and is valid for all modes $0 \leq k \leq \infty$ and due to the convolution integral all these modes are coupled. Thus, the solution of the NSE represents an enormous task and in order to cut out unnecessary work, the physically important quantities must be identified.

1.2 Statistical Methods

For practical purposes, it is the large scale effects of the flow that are most important and hence it is common practice to introduce a decomposition of the velocity field into a mean and a fluctuating part, thus

$$u_\alpha(\mathbf{k},t) = \overline{u}_\alpha(\mathbf{k},t) + u'_\alpha(\mathbf{k},t). \tag{1.11}$$

An averaging procedure is required to produce the mean part of the velocity field, usually the ensemble average over all possible solutions of the NSE given certain initial conditions, denoted by Dirac brackets $\langle\rangle$. This average is called the turbulent average and it is assumed that turbulence is ergodic. Hence, any trajectory in phase space of a single initial state will come arbitrarily close to all points in infinite time and thus, infinite time averages are equivalent to ensemble averages. Hence it is proper to relate experimental time series averages to the results of theory which will predict for ensemble averaged quantities. The ensemble is generated by considering different initial or boundary conditions, e.g. varying the precise form of the stirring forces while maintaining the same statistical properties.

The question of ergodicity in turbulence remains open. However, for the Euler equation it has been shown[1] that the non-linear term leads to the development of mixing behaviour which implies ergodicity. The Euler equation is the NSE with

the viscosity set to zero. It represents an interesting test case for ideas in turbulence since it illustrates purely the non-linear effects, there being no dissipation. However, it must be stressed that the behaviour of an Euler flow is not the $\nu_0 \to 0$ or $Re \to \infty$ limit of an NSE flow. The energy spectra are vastly different, this will be discussed in the next section.

As turbulence is a highly complex problem, the simplest form will be tackled, i.e. homogeneous, isotropic, stationary turbulence. The description of the flow as homogeneous and isotropic always relates to the behaviour under the turbulent average of the velocity field. The turbulence is homogeneous which means that the velocity field is homogeneous under this average,

$$\overline{u}_\alpha(\mathbf{k},t) =< u_\alpha(\mathbf{k},t) >= 0. \tag{1.12}$$

Using this average on equation(1.8), the equation for the behaviour of the mean field is

$$\begin{aligned}\left[\frac{\partial}{\partial t} + \nu_0 k^2\right]\overline{u}_\alpha(\mathbf{k},t) &= M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\{\overline{u}_\beta(\mathbf{j},t)\overline{u}_\gamma(\mathbf{k}-\mathbf{j},t) + < u'_\beta(\mathbf{j},t)u'_\gamma(\mathbf{k}-\mathbf{j},t) >\} \\ &\quad + < f_\alpha(\mathbf{k},t) > . \end{aligned} \tag{1.13}$$

The $< u'u' >$ term is the contribution of the fluctuating part to the mean flow and is often modelled as a linear function of $\overline{u}_\alpha(\mathbf{k},t)$. The equation above can then be rewritten as

$$\left[\frac{\partial}{\partial t} + \nu_T k^2\right]\overline{u}_\alpha(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\overline{u}_\beta(\mathbf{j},t)\overline{u}_\gamma(\mathbf{k}-\mathbf{j},t) + < f_\alpha(\mathbf{k},t) >, \tag{1.14}$$

where ν_T is called the eddy(or effective) viscosity whose form is modelled (e.g. by a mixing length model).

The above paragraphs introduce two fundamental ideas that are worth reiterating. Firstly, our method is to minimise the difficulty of the problem by only seeking exact solutions for the physically significant degrees of freedom (small wavenumber

modes of the velocity field), here the mean field. Secondly, the fluctuating modes are to be dealt with only on the average as they effect the mean.

The second point is, however, much more complex than might first appear as an exact solution for the second moment of the fluctuating field depends on the third moment and the third moment on the fourth moment and so on. The ensuing equations are called a moment hierarchy and this problem is known as the hierarchy problem. Overcoming this difficulty forms a central part of solving turbulence and has usually taken the form of a truncation of the hierarchy.

The solution requires a knowledge of the probability distribution of the velocity field. Here is one facet of the difficulty of turbulence, which is its dissipative nature. Since we are far from equilibrium, the probability distribution is unlikely to be a simple Gaussian and the appearance of coherent structures in the flow makes less plausible any guesses based on purely random distributions. However, formulating the problem in terms of generating function of the probability distribution[2] brings us no nearer a solution since no solution has yet been found to the equation for that generating functional.

1.3 The Energy Picture

The most important moment of the velocity field is the second moment which is related to the energy of the turbulence per unit mass through the expression,

$$2E = 3\langle u^2 \rangle. \tag{1.15}$$

Now, introducing some useful notation, the second moment can be expressed as follows

$$\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{k}',t')\rangle = \delta(\mathbf{k}+\mathbf{k}')Q_{\alpha\beta}(\mathbf{k};t,t'), \tag{1.16}$$

for homogeneous turbulence (homogeneous refers to the behaviour of the fluid under the average). Since the turbulence to be considered is also isotropic and stationary then this implies

$$\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{k}',t')\rangle = \delta(\mathbf{k}+\mathbf{k}')D_{\alpha\beta}(\mathbf{k})Q(k,t-t'). \tag{1.17}$$

Thus, the energy is, for a single time correlation,

$$\begin{aligned} 2E &= tr\int d^3k Q_{\alpha\beta}(\mathbf{k}) \\ &= tr\int_0^\infty q(k)k^2dk\int D_{\alpha\beta}(\mathbf{k})d\Omega_k \\ &= \int_0^\infty 8\pi k^2 q(k)dk. \end{aligned} \tag{1.18}$$

This defines the energy per wavenumber as

$$E(k) = 4\pi k^2 q(k). \tag{1.19}$$

Now, to obtain the equation for $E(k)$ multiply the NSE by $u_\beta(-\mathbf{k},t)$ and average, then set $\beta = \alpha$ and sum over the indices,

$$\left[\frac{d}{dt} + 2\nu_0 k^2\right] E(k,t) = T(k,t) + W(k,t), \tag{1.20}$$

where W(k,t) represents the work done by the stirring forces on the velocity field and the term $T(k,t)$ represents the complex non-linear energy transfer term which it can be shown conserves energy globally[3]. Thus, integrating we obtain the expected balance equation

$$\frac{dE}{dt} + \int_0^\infty 2\nu_0 k^2 E(k,t)dk = W(t), \tag{1.21}$$

which defines $W(t)$ and shows how the input of energy by the stirring forces is dissipated by viscosity. Stationarity is the condition where the input of energy exactly balances the output of energy to heat, i.e.

$$W(t) = \epsilon = \int_0^\infty 2\nu_0 k^2 E(k,t)dk \tag{1.22}$$

The term ϵ is called the dissipation rate of the turbulence.

This level of description is too vague since it involves the removal of detail concerning the behaviour of the individual modes which in some cases may be desirable. So, instead, the energy per wavenumber or energy spectrum will be considered. As yet, there is no solution of the NSE which yields the high Reynolds number form of the energy spectrum, however, by making certain assumptions it is possible to identify simply the form of the spectrum under certain conditions.

The cascade picture of energy transfer views the process of turbulence as transferring energy from the low wavenumber modes (where the stirring forces are pumping energy into the flow) to the high wavenumber modes (where viscous dissipation drains the energy out as heat). Already, this identifies two regions of k space which I will call the *energy-containing range* and the *viscous range*, respectively. (The low wavenumber range is called the energy-containing range as most of the total energy in the flow resides there.In this range, the peak of the energy spectrum coincides with the region of action of the stirring forces.)

In order to further clarify the picture, some measure of the appropriate wavenumbers for these regions is required. In the case of the energy-containing range the physics is dominated by specific elements of the problem. That is to say, the stirring forces and the boundary conditions(or geometry of the problem) make themselves most strongly felt in this region and as such its dynamics are *non-universal*, in the sense of changing from one problem involving turbulence to another. Thus, the wavenumber of the peak of the energy spectrum, k_p will suffice as characteristic of the range.

For the viscous range, the argument is more complex and more informative since here we have a region whose dynamics being dominated by viscous dissipation are *universal* to any turbulence problem. In this region it is argued, there are only two important parameters the viscosity and the dissipation rate. Hence, on

dimensional grounds, the wavenumber(inverse length) scale associated will be

$$k_d = (\frac{\epsilon}{\nu_0^3})^{1/4}, \tag{1.23}$$

which is called the Kolmogorov wavenumber. This argument depends also on the interactions between modes being local in nature. This seems fairly self-evident in the case of the viscous range as damping will be randomising the action of the modes, destroying any long-range correlations.

Now, in high Reynolds' number flows, there will exist a third region according to Kolmogorov[4]. In this range of wavenumbers the inertial term dominates the dynamics, as is to be expected for a high Reynolds' number flow.Of course, there must still exist the viscous and energy-containing ranges. Kolmogorov hypothesises that the interactions between modes are local (i.e. strong between similar length scales) and thus the energy spectrum in the region where inertial interactions prevail will depend only upon the rate energy is fed in to these scales and the wavenumber associated with that scale. Therefore, by dimensional analysis, we obtain for the energy spectrum

$$E(k) = \alpha \epsilon^{2/3} k^{-5/3}. \tag{1.24}$$

This is the well-known Kolmogorov spectrum or '-5/3' power law, where the constant α is called the Kolmogorov constant. (Kolmogorov's hypotheses actually relate to the probability distribution of the velocity modes but the practical effect is that given[5].)

The range of wavenumbers where the above power law will prevail is called the *inertial range* and its extent will depend on how well separated are k_p and k_d. As the Reynolds' number increases, by letting the viscosity decrease, the Kolmogorov wavenumber gets larger and thus the inertial range extends(k_p will not alter significantly). In the limit of $\nu_0 \rightarrow 0$ we have an infinite Re, dissipation acts only at $k = \infty$ and the inertial range is infinite.

The existence of a Kolmogorov spectrum in high Re turbulence is well verified experimentally[6] although the value of the constant, α, remains open to question. The only doubt about the value of the power law exponent arises when fine-scale intermittency effects[1] are considered although the correction to the spectrum is very small, on the order of 4%[8]. Monin and Yaglom[9] carry out an extensive survey of the experimental literature on the Kolmogorov constant and declare it has a value of 1.5 with, at most, an error of 20%.

Returning to the Euler equation mentioned earlier, it provides an interesting comparison with the NSE and sheds some light on the energy transfer process. In an Euler flow, there is no dissipation and consequently the inertial term creates energy equipartition(i.e. equal energy in each mode) and thus $E(k) \propto k^2$. Obviously, this is totally wrong for an NSE flow. However, it creates a picture of the division of tasks among the various terms in the NSE where the stirring forces pump in the energy, the inertial terms attempt to spread it evenly throughout the spectrum but the viscous dissipation drains it out at high wavenumbers. So the ultimate effect of the inertial terms is to be the transfer mechanism for energy down to the scales where it is most effectively dissipated.

1.4 Large Eddy Simulation

The remarkable increase in computational power available to a scientist tackling turbulence has led to a change in the perception of the problem. It is now possible to directly simulate the NSE on a computer to a moderate Re (enough for a small inertial range). This has changed the aim of turbulence theory to an extent, in that it is now no longer necessary to hope to describe all the modes active in a turbulent flow(this also remains outside the scope of computational power)

[1]These are due to the discovery by Batchelor and Townsend[7] that dissipation is concentrated in certain areas of the flow and thus is not constant.

but, instead, a combination of the two approaches is proposed with the computer simulating as many modes as it can while theory provides for the effect of those neglected.

This is called a Large Eddy Simulation where only the smaller wavenumber modes are simulated using a modified NSE. It can be viewed as a grid placed over the flow with only the scales above the unit length of the grid being simulated. The effect of the small scales, thus neglected, is treated by introducing an effective viscosity which accounts for the effect of these 'subgrid' scales on the 'supergrid' or 'explicit' scales. Thus, if the effective viscosity is accurate the LES gives an precise description of the large scale behaviour of the velocity modes.

Considering this explicitly, if one starts with the NSE and then divides the velocity field into two parts as follows,

$$u_\alpha(\mathbf{k}) = \begin{cases} u_\alpha^-(\mathbf{k}) & \text{for } 0 \le k \le k_c \\ u_\alpha^+(\mathbf{k}) & \text{for } k_c \le k \le k_0. \end{cases} \tag{1.25}$$

k_c is the cut-off wavenumber between the resolved scales and the subgrid scales. This is called a Fourier cut-off filter and is one of several different ways in which the velocity field can be decomposed for a Large Eddy Simulation. The modified NSE for the supergrid scales now looks like

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha^-(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j (u_\beta^-(\mathbf{j},t) + u_\beta^+(\mathbf{j},t))(u_\gamma^-(\mathbf{k}-\mathbf{j},t) + u_\gamma^+(\mathbf{k}-\mathbf{j},t)) + f_\alpha(\mathbf{k},t). \tag{1.26}$$

The simulated equation for the explicit scales will ultimately look like

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2 + \nu(k|k_c)k^2\right] u_\alpha^-(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j u_\beta^-(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t) + f_\alpha(\mathbf{k},t). \tag{1.27}$$

The stirring force will remain unchanged, provided it falls completely within the resolved region.

The new element in the equation above is the introduction of an effective viscosity term which accounts for all the terms involving $\mathbf{u}^+$'s coupled to the $\mathbf{u}^-$'s. This term represents the transfer of energy down the cascade from the large to the small scales and dissipation, a process which in high Re flow will be via the inertial range. Finding an analytical form for the drain of energy from the explicit scales ($k \leq k_c$) to the subgrid scales ($k \geq k_c$) is called the 'subgrid-modelling problem'.

The first attempts to solve the subgrid modelling problem were based on a relation between the subgrid stress and the strains in the explicit scales. This closure was due to Smagorinsky[10] and introduces the concept of an eddy viscosity as the proportionality constant between these stresses and strains. This is done on a phenomenological level as are most of the subgrid models. However, in this work I will attempt to find a theoretical framework for posing this problem exactly.

Chapter 2

The application of Renormalisation Group methods to fluid turbulence

2.1 Introduction

I will now outline the application of Renormalisation Group(RG) methods to turbulence. Firstly, I will discuss the nature of RG and note the similarities and differences between the existing formalism and the requirements of a method which tackles the turbulence problem. Secondly, the theoretical approach to the turbulence problem will be discussed beginning with renormalised perturbation theory and then RG which can be applied either via the theory of strongly stirred hydrodynamics or the recursion relation approach. The motivation for the approach detailed in chapter 3 will appear as the problems observed in other methods are examined.

2.2 Background

Renormalisation is a method used in quantum physics for the removal of certain singularities which appear in the continuum limit. In a more general way, this can be looked upon as redefining of interaction strengths so as to write the Hamiltonian in a more tractable form. A classic example is in the interaction of an electron in a plasma with the cloud of charge that surrounds it. Debye and Huckel realised that by redefining the charge of the electron they could continue to use the Coulomb-like potential. This illustrates how a problem with several degrees of freedom can be viewed in a quasi-single particle way using a renormalisation. It is also interesting to note that this involves a consideration of the average effect of the cloud on the single particle as a basic approximation.

The Renormalisation Group is an attempt to systematise this procedure for problems involving large numbers of strongly interacting degrees of freedom. The fundamental assumption is that interactions are local in space. The theory of this was formalised by Wilson and is reviewed in an excellent paper by Wilson and Kogut[11].

This approach is based on a k-space decimation of the degrees of freedom in the problem. The initial assumption is that we are interested only in the large scale(low wavenumber) behaviour of some assembly of modes. Consequently, a way has to be sought to describe the effect of the small scale modes on the larger scales.

Wilson introduces a second cut-off at k_1, apart from the ultraviolet one at k_0 and all the modes within $k_1 \leq k \leq k_0$ are then averaged over. Their effect is to increment the coupling constant between large scale modes and also to introduce new couplings (at higher orders of complexity). The variables are then rescaled so that the Hamiltonian for these new modes (i.e. those of the larger scales)

looks the same as before with the exception of these new terms. The procedure is repeated until the transformed Hamiltonian is the same as the old Hamiltonian at which point the iteration is said to have reached a fixed point. Introducing the transformation τ as the RG transformation then, starting with the initial Hamiltonian, $\mathcal{H}_0$, we can write this procedure as,

$$\tau(\mathcal{H}_0) = \mathcal{H}_1 \tag{2.28}$$

until finally

$$\tau(\mathcal{H}_{n-1}) = \mathcal{H}_n = \mathcal{H}_{n-1} \tag{2.29}$$

which is the fixed point Hamiltonian, $\mathcal{H}^*$.

The theory progresses provided a fixed point can be found and often this requires the introduction of perturbation theory - in the classic case (Wilson and Fisher[12]) the perturbation is in the space dimensionality of the assembly.

The analogies with turbulence are clear:- the existence of a large number of strongly coupled modes in the inertial range and the belief that these will only be coupled locally in k-space; the desire to have a large scale scale description of the problem as solution. However, there is no Hamiltonian description of a dissipative problem such as turbulence and the probability distribution of the turbulent velocity field is unknown and certainly not Gaussian. This second point provides extra difficulties in dealing with average effects of small scales on large scales.

2.3 The theoretical approach

The turbulence problem will be considered as a problem in solving the Newtonian equation for the dynamics of the fluid. Restating the equation of motion for

incompressible fluid flow, the NSE, in its spectral form,

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha(\mathbf{k},t) = \lambda\, M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3\mathbf{j} u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t) + f_\alpha(\mathbf{k},t), \qquad (2.30)$$

where $M_{\alpha\beta\gamma}(\mathbf{k})$ and $D_{\alpha\beta}(\mathbf{k})$ are as defined in chapter 1. Here, λ is a book-keeping parameter, which is subsequently set equal to unity.

For a velocity field with zero mean, which a homogeneous field must have, the principal statistical quantity of interest is the energy spectrum. From chapter 1, it can be defined in terms of the second moment of the velocity field as follows:

$$E(k,t) = 2\pi k^2 \left\langle u_\alpha(\mathbf{k},t) u_\alpha(-\mathbf{k},t)\right\rangle . \qquad (2.31)$$

It is conventional to specify the ensemble by choosing a stirring force with Gaussian statistics, and a second moment defined by

$$\langle f_\alpha(\mathbf{k},t) f_\alpha(-\mathbf{k},t')\rangle = W(k)\delta(t-t'). \qquad (2.32)$$

A general requirement for $W(k)$ is that it is peaked near k=0, so that universal behaviour may develop at higher wavenumbers due to non-linear transfer. This is also a practical approach as in most real situations the stirring will be done on the large length scales.

Any attempt to solve the problem will begin with the acknowledgement that it is the presence of the non-linear term that causes the difficulty. Hence, in true scientific style we begin by pretending it is not there. The NSE without the non-linear term is called the Stokes equation and it yields a solution known as the viscous solution. The next refinement in this process is to try and perturb about this solution. In other words, it is assumed that the non-linear term is small(which it is not) and expand in λ about the viscous solution. It is well known that such a perturbation expansion leads to a divergent series. We show this in an abbreviated notation as follows:

$$u_\alpha(\mathbf{k}) \;=\; u_\alpha^{(0)}(\mathbf{k})$$

$$+\lambda G_0(\mathbf{k})M_{\alpha\beta\gamma}(\mathbf{k})\int d^3\mathbf{j}\, u_\beta^{(0)}(\mathbf{j})u_\gamma^{(0)}(\mathbf{k}-\mathbf{j})$$
$$+\lambda^2 G_0(\mathbf{k})M_{\alpha\beta\gamma}(\mathbf{k})\int d^3\mathbf{j}\ \times$$
$$[u_\beta^{(0)}(\mathbf{j})G_0(\mathbf{k}-\mathbf{j})M_{\gamma\rho\epsilon}(\mathbf{k}-\mathbf{j})\int d^3\mathbf{l}\, u_\rho^{(0)}(\mathbf{l})u_\epsilon^{(0)}(\mathbf{k}-\mathbf{j}-\mathbf{l})$$
$$+u_\gamma^{(0)}(\mathbf{k}-\mathbf{j})G_0(\mathbf{j})M_{\beta\rho\epsilon}(\mathbf{j})\int d^3\mathbf{l}\, u_\rho^{(0)}(\mathbf{l})u_\epsilon^{(0)}(\mathbf{j}-\mathbf{l})]$$
$$+O(\lambda^3), \tag{2.33}$$

where

$$u_\alpha^{(0)}(\mathbf{k}) = D_{\alpha\beta}(\mathbf{k})G_0(\mathbf{k})f_\beta(\mathbf{k}) \tag{2.34}$$

and $G_0 = (i\omega + \nu_0 k^2)^{-1}$ is called the zero-order(*i.e.* unrenormalised) propagator. In eq.(2.33) the integral sign denotes integration over all the running variables. Higher order terms are obtained by iterating the NSE.

In the past, this problem has been treated by summing certain classes of terms in order to achieve renormalisation of the viscous propagator and the two-point correlation. The result has been a variety of Renormalised Perturbation Theories(RPT) [13, 14, 15, 16]. While these theories have had qualified success[17, 18, 19] in predicting measurable quantities, it is still necessary to truncate a series of uncertain convergence properties.

The Renormalisation Group (RG) approach offers a new way of tackling the problem of reducing the number of degrees of freedom necessary to describe turbulent flows at large Reynolds numbers. There are two different ways of applying RG to turbulence:

(a) derive differential relations for the renormalised quantities and seek a fixed point in the coupling-constant space; and

(b) derive a recurrence relation, which maintains form-invariance of the dynamical equation, and seek a fixed point for the renormalised viscosity.

Method (a) was pioneered by Forster *et al.*[20] (to be referred to as FNS hereafter), who considered the effect of three different stirring force spectra, defined for wavenumbers in the range $0 \leq k \leq \Lambda$, where the cut-off Λ is chosen to be small enough to exclude inertial effects. The velocity field was split into two components by a cut, $\Lambda exp(-l)$, and then a perturbation series in the non-linear coupling developed for the effect of the high wavenumber modes on the low wavenumber modes. The series is an expansion about the viscous solution and consequently, depends on the statistics of the forcing function. The viscosity was renormalised by a contribution from this truncated perturbation series in a narrow band of wavenumbers at k=Λ.

By choosing $W(k) = W_0 k^{4-d-\tilde{\epsilon}}$, the problem can be tailored to resemble analytical forms of the Ising model[21], with an upper critical dimension d=4. A dynamic RG iteration is performed where the bandwidth within which the perturbation series is performed is shrunk, by letting $l \rightarrow 0$. Differential relations for the renormalised forms of viscosity, forcing spectrum and coupling strength are obtained. A fixed point in the coupling strength is found. The calculations are performed in the limit of small k and are only asymptotically valid. It is found that the higher order terms in the small wavenumber modes are irrelevant variables under this scheme (i.e. the coupling parameters with which they appear in the new dynamical equation vanish as the iteration approaches the fixed point.)

The dynamics of the problem are defined by the choice of forcing spectrum and the results only valid in the long wavelength limit. Thus, it is questionable whether this work can have any relation to real turbulence at all since the independence of the results for a turbulent flow from the forcing or boundary conditions have long been held to be strong evidence in favour of universal behaviour. This artificiality is compounded by the restriction of the range of wavenumbers to exclude the inertial range and the assumption that the forcing occurs strongly over the whole range of modes considered, which in a real flow would not be the case.

DeDominicis and Martin[22] confirmed these results using field-theoretic methods (introducing a generating functional). However, this work is subject to the same restrictions as above for FNS. They also noted that the choice $d = 3$, $\tilde{\epsilon} = 4$ would yield the Kolmogorov power law. They express concern about the consequent infra-red divergence which seems inappropriate to any real problem where the stirring and energy spectra will be finite at $k = 0$. This choice of stirring spectrum does not give a finite excitation level, as can be seen from the following. The rate of doing work on the system, W , is given in this case by

$$\begin{aligned} W &= \int_{k_{min}}^{k_{max}} 4\pi k^2 W(k) dk \\ &= 4\pi W_0 \int_{k_{min}}^{k_{max}} k^{-1} dk \\ &= 4\pi W_0 ln(k_{max}/k_{min}). \end{aligned} \tag{2.35}$$

Clearly, W is log-divergent as $k_{max} \rightarrow \infty$, which is suggested as a valid limit for the theory. For stationarity, this must imply an infinite rate of dissipation, with obvious consequences for the inertial-range spectrum as given by eq.(1.24).

On the other hand, if one follows DeDominicis and Martin[22], and forces the calculations to have a finite rate of dissipation, by arbitrarily setting $W_0 = \epsilon$, then it can be shown, from eq.(2.35), that the excitation must be restricted to a range of wavenumbers such that k_{max}/k_{min}=1.083. This emphasises the unreal nature of this model for real flows, where the inertial range can extend over many decades dependent on the Reynolds' number.

Yakhot and Orszag[23] have proposed that the conjecture of DeDominicis and Martin[22], though outside the range of validity of the FNS theory, could, nevertheless, be used for practical applications. In order to do this they postulated that there is a 'correspondence principle', which implies that numerical results obtained in an *ersatz* inertial range produced by strong stirring will be the same as those of a turbulent inertial range. The resulting good agreement which they find between a wide range of calculated and experimental values is then given as

justification for this principle.

In order to have a finite energy input, Yakhot and Orszag invoke the relation,

$$\frac{\gamma}{\alpha^2} = 0.1904, \tag{2.36}$$

originally derived by Edwards[24] from an RPT energy balance equation, now derived from the Eddy-Damped Quasi-normal Markovian equations which they have obtained from the RG analysis[25]. The expansion parameter in this case is a renormalised Reynolds' number which is proportional to $\tilde{\epsilon}^{1/2}$. They make the assumption that results derived in the limit as $\tilde{\epsilon} \to 0$ are still applicable when $\tilde{\epsilon} = 4$. There is, as yet, no proof of the validity of the truncation of series that this allows.

It has been pointed out by Kraichnan[26] that these attempts are doomed, as are RPT theories, due to unknown analyticity properties of the series developed. Kraichnan also notes that the treatment of the low wavenumber modes by an eddy viscosity does not agree with the picture of convection and straining by the large scale structures on each other, which does not allow scale separation.

An interesting development was made recently by Avellaneda and Majda[27] who have set up a model equation for passive scalar transport which has exact renormalisation of the effective diffusivity. The results are for the long wavelength region. They have shown that the results for this model are the same for the assumptions of Gaussian and non-Gaussian statistics for the turbulent velocity field. It is also suggested that this model may be used as a test for other theories since it is exact.

In the second RG approach, method (b), the field under investigation is again split into two parts: small wavenumbers (slow modes) and large wavenumbers (fast modes). Then, the effect of the large wavenumbers on the small wavenumbers is renormalised into an effective viscosity coefficient. The purpose of the renor-

malisation is to maintain the form-invariance of the equation of motion, under an iteration which progressively averages out bands of high wavenumber modes. The existence of a fixed point in the recursion relation for the renormalised viscosity then indicates the presence of universal scaling behaviour (an inertial range).

At this point we note, as an aside, that in method (a), the non-linear coupling is made small by balancing viscous and stirring terms, while in method (b), the linearisation scheme is carried out in the high-k region where the non-linearity is physically small. The other major differences between this approach and that of method (a) are that the wavenumber region dealt with is that of the energy-containing and inertial wavenumbers; the bandwidth remains finite in method (b) and the results are valid for all k in the low-wavenumber region left, once the fixed point is reached. It is to be emphasised that method (b) cannot be implemented by perturbation theory about the viscous solution, nor should it, since it depends fundamentally on the dominance of the inertial interactions.

The first attempt to apply the finite bandwidth approach in a fluid mechanics problem was by Rose[28]. Here, he attempted to evade the difficulty of the strong non-linearity by considering the explicitly linear problem of passive convection of a scalar field (ϕ, say) by a turbulent velocity field. He used perturbation theory, but instead of the triple non-linearity (uuu), he was faced with an analogous term (ϕuu), which he treated as part of a new scalar diffusion equation. This new equation exhibits form-invariance under RG transformation, and a fixed point for the renormalised diffusivity is obtained. But, of course, it still involves an arbitrary truncation, which has to be repeated in every cycle of the iteration, in order to preserve the form-invariance.

This approach is the starting point for two other lines of work(that of Iterative Averaging and Recursive Renormalisation Group). The finite bandwidth of the wavenumbers to be removed at each iteration leads to a recursion relation involving the solution of numerical integrals to find the fixed point for any k in the low-

wavenumber band. This differs from the analytic results of method (a), which lead to an effective viscosity independent of wavenumber.

The method of Iterative Averaging(IA) was proposed by McComb as an alternative to perturbation theory[29]. This approach started as a short time average over the velocity field[30, 31]. However, it was found that a more natural viewpoint was to be found in an average which acted only on modes in part of k-space. This local (in k-space) average is made the defining operation which divides the field into high-k and low-k parts. Iteration and rescaling led then to a fixed point, corresponding to the renormalised (eddy) viscosity. Crucially, the IA procedure recognised the important nature of the average that is carried out in the formulation. The average must only act upon the high wavenumber modes since it is the behaviour of the smaller wavenumber modes that is sought. It has been demonstrated that the underlying equivalent of the perturbation expansion in this method is a moment expansion in the band of eliminated modes.

It is this work that represented the starting point for the work given in this thesis, answering the outstanding questions from[29]. However, the redevelopment has been extensive so that detailed discussion would only lead to confusion.

The Recursive Renormalisation Group approach of Zhou *et al.*[32] follows McComb and Shanmugasundaram[33] where Rose's theory is applied to the velocity field problem. There are differences of viewpoint between this work and the course to be followed later which it may prove useful to discuss.

To start with, we summarise their procedure as follows:

1. Split the velocity field into subgrid and supergrid fields.
2. Introduce an ensemble average over only the subgrid modes.
3. Using a perturbation series in the coupling strength of the non-linear term,

solve for the contribution of the subgrid scales to the supergrid scales equation as an increment to the viscosity, to lowest non-trivial order.

4. Truncate the moment hierarchy arbitrarily by ignoring third order moments in the subgrid field.

5. Rescale and repeat this procedure until a fixed point in the effective viscosity is found. (Maintaining consistency of the truncation and form-invariance of the dynamic equation for the supergrid modes)

The first point concerns the process of taking averages. There is no way to create this ensemble average in a simple fashion, say as a combination of a filter and the global ensemble average. Under such an average, the cross-terms involving the product of fast and slow modes do not all vanish. This requires a special kind of average, with the implication that one has to define the non-trivial relationship between local and global averages. Zhou *et al.* do not address this question and it is this point that is crucial in the formulation using a conditional average(see next chapter).

The second point carries over from Rose's work and is the necessity of truncating the power series and the moment hierarchy arbitrarily and with no measure of the implications of these approximations. This motivates the need for a picture in which the neglect of these terms can be justified. The theory must be cast in a form where it is natural to regard these as less important terms, which is generally expected but still unproved.

Zhou *et al.* attach great importance to the occurrence of a 'cusp' near the cut-off between high and low wavenumbers. In their results[34], the cusp appears for one value of the bandwidth parameter (f=0.7) but not at another (f=0.6), provided the effect of the triple non-linearity is included. Our first reaction to this trend is to observe that the cusp is not actually a qualitative feature of their

results, appearing as it does for f=0.7 and disappearing at f=0.6. Our experience of computation in this area leads us to doubt the presence of non-trivial physical effects in going from f=0.7 to f=0.6. This raises questions about the accuracy of their numerical procedures. Due to the re-scaling, the problems associated with cancellation of separately divergent terms as $k \to 0$, can also affect the results in the neighbourhood of the cut-off.

On this point, our second observation is that it is misleading to suggest that the presence, or otherwise, of a cusp is a criterion for judging the merits of a particular theory. It is, of course, intuitively plausible that there should be a build-up in renormalised viscosity at the cut-off, due to the elimination of modes above it. However, the evidence quoted for its existence is less than impressive. The Test Field Model of Kraichnan[35] gives a cusp due to an integrable singularity at $k = 0$; and not because of local interactions across the cut.(This is due to the application of the Kolmogorov spectrum over the entire spectrum, including the origin.) The simulations of Chollet and Lesieur[36] and Domaradzki *et al.*[37] exhibit this behaviour for precisely the same reason. Further, it has been demonstrated by Leslie and Quarini[38] that the cusp can be removed by a more realistic choice of energy spectrum which vanishes as $k \to 0$. It is interesting to note that this choice corresponds to the case where the whole inertial range has been removed and only the energy-containing range remains. This is precisely the situation desired once the fixed point for the inertial range is found.

Chapter 3

A Renormalisation Group method for deriving the Effective Viscosity using a Conditional Average

3.1 Introduction

The problem tackled here is to find an expression for the effective viscosity to be used in a Large Eddy Simulation. From this, a value for the Kolmogorov constant can be found and its dependence on parameters in the method qualitatively explained.

As stated in the previous chapter, the work described has developed from a series of papers[30, 31, 29] which attempted to solve this problem by introducing an average over only the high wavenumber modes. However, the nature of the average remained unclear as did the reasons for the breakdown of the theory with varying width of the band of wavenumbers averaged out. The theory was then recast so as to make more explicit the precise nature of the average and a more quantitative approach to the problem developed.

3.2 Derivation of the Effective Viscosity

3.2.1 Rewriting the Navier-Stokes Equation

Consider the turbulent velocity field in wavenumber space $u_\alpha(\mathbf{k},t)$, on the interval $0 \leq k \leq k_0$, with k_0 being defined through the dissipation integral

$$\epsilon = \int_0^\infty 2\nu_0 k^2 E(k)dk \doteq \int_0^{k_0} 2\nu_0 k^2 E(k)dk \tag{3.1}$$

where ϵ is the dissipation rate and E(k) is the energy spectrum. This definition ensures that k_0 is of the same order of magnitude as the Kolmogorov dissipation wavenumber.

In order to introduce the renormalisation group approach, we divide up the velocity field at $k = k_1$ in the following way

$$u_\alpha(\mathbf{k}) = \begin{cases} u_\alpha^-(\mathbf{k}) & \text{for } 0 \leq k \leq k_1 \\ u_\alpha^+(\mathbf{k}) & \text{for } k_1 \leq k \leq k_0, \end{cases} \tag{3.2}$$

where k_1 is defined by

$$k_1 = (1-\lambda)k_0, \tag{3.3}$$

with the bandwidth parameter λ satisfying the condition $0 \leq \lambda \leq 1$. The modes $\mathbf{u}^+$ are those in the band to be averaged. Note that when $\lambda = 1$ then $k_1 = 0$ and we are doing Reynolds averaging (over all length scales) and when $\lambda = 0$ no modes are averaged. The superscripts '+' and '-' denote the range of wavenumbers allowed for a mode; this has been found to be a convenient notation, avoiding the use of θ functions.

In principle, the renormalisation group approach now involves three stages:

A. Solve the Navier-Stokes equation (NSE) on $k_1 \leq k \leq k_0$. Substitute that solution for the mean effect of the high-k modes into the NSE on $0 \leq k \leq k_1$. This results in an increment to the viscosity: $\nu_0 \rightarrow \nu_1 = \nu_0 + \delta\nu_0$.

B. Rescale the basic variables, so that the NSE on $0 \leq k \leq k_1$ looks like the original Navier-Stokes equation on $0 \leq k \leq k_0$.

C. Repeat A and B until the scaled viscosity no longer changes (i.e. a fixed point is reached).

This procedure is appealingly simple and has a clear physical interpretation. But, as is well known, it has not proved easy to put it into practice in the turbulence problem. The system is dissipative and therefore there is no simple Hamiltonian formalism on which to rely; consequently, the RG transformation acts so as to maintain *form invariance* of the equation of motion. The transport coefficient must be renormalised at each iteration to account for the mean effect of the $\mathbf{u}^+$ modes. However, it is clear that at each iteration of procedures A and B the effective viscosity increases as more wavenumbers are averaged out. The fixed point exists, therefore, in a scaled form of the viscosity only. A rational way of doing this will be introduced using an approximation in which the bandwidth parameter, λ, plays the part of a small parameter.

It is necessary to introduce a conditional average which averages out the effect of high-k modes, while keeping the $\mathbf{u}^-$ constant. That is, it is an average over a subensemble of solutions of the NSE in which realisations are chosen from the normal turbulent ensemble to hold $\mathbf{u}^-$ constant. Ergodicity and the chaotic nature of turbulence generate the realisations for the average with all possible solutions for $\mathbf{u}^+$ in the NSE, with $\mathbf{u}^-$ as boundary conditions. We represent the average by the operator $A[\mathbf{u}^+|\mathbf{u}^-]$ and denote its effect on the first shell of wavenumbers to be eliminated by $\langle\ \rangle_0$, thus

$$A[\mathbf{u}^+|\mathbf{u}^-]u_\alpha u_\beta \ldots u_\gamma = \langle u_\alpha u_\beta \ldots u_\gamma \rangle_0. \tag{3.4}$$

It then follows from the definition, that this operator, when acting on the low-k modes, has the following properties

$$\langle u_\alpha^-(\mathbf{k}) \rangle_0 \ = \ u_\alpha^-(\mathbf{k}), \tag{3.5}$$

$$\langle u_\alpha^-(\mathbf{j}) u_\beta^-(\mathbf{k}-\mathbf{j})\rangle_0 \;=\; u_\alpha^-(\mathbf{j}) u_\beta^-(\mathbf{k}-\mathbf{j}). \tag{3.6}$$

We now wish to evaluate averages of this kind over the high-k modes and express them in terms of global mean quantities. The problem we face is that the $\mathbf{u}^+$ field is not independent of the $\mathbf{u}^-$ field which we are holding constant. The two fields are of course coupled together through the non-linear term in the Navier-Stokes equation. We tackle this difficulty by writing the high-k modes in terms of a new field $\mathbf{v}^+$, thus

$$u_\alpha^+(\mathbf{k},t) = v_\alpha^+(\mathbf{k},t) + \Delta_\alpha^+(\mathbf{k},t) \tag{3.7}$$

Here $\mathbf{v}^+$ is a field of the same general type as $\mathbf{u}^+$. It has zero mean and is homogeneous, isotropic and stationary under *global* averaging, thus:

$$\langle v_\alpha^+(\mathbf{k},t)\rangle = 0, \tag{3.8}$$

$$\langle v_\alpha^+(\mathbf{k},t) v_\beta^+(\mathbf{k}',t)\rangle = Q_v^+(k) D_{\alpha\beta}(\mathbf{k}) \delta(\mathbf{k}-\mathbf{k}') \tag{3.9}$$

However, the essential feature of $\mathbf{v}^+$ is that it is not coupled to the $\mathbf{u}^-$ modes. The introduction of a plausible physical variable for $\mathbf{v}^+$ is crucial to calculations using this method but will be dealt with later. Thus, through equation (3.7) we introduce the function $\mathbf{\Delta}^+$ to take account of such mode coupling. It vanishes under global averaging as from equations (3.7) and (3.8), we have

$$\langle \Delta_\alpha^+(\mathbf{k},t)\rangle = 0. \tag{3.10}$$

With all these points in mind, we complete our specification of the new field by stating its properties under *conditional* averaging as:

$$A[\mathbf{u}^+|\mathbf{u}^-]\mathbf{v}^+(\mathbf{k},t) = \langle \mathbf{v}^+(\mathbf{k},t)\rangle_0 = \langle \mathbf{v}^+(\mathbf{k},t)\rangle = 0. \tag{3.11}$$

The $\mathbf{v}^+$ field is the part of the $\mathbf{u}^+$ field that is independent of $\mathbf{u}^-$ and consequently, the conditional average has the same effect as the global average. I will show that

the Δ^+ field, which contains all the awkward coupling information, can be treated as a correction to the main effect of the $\mathbf{v}^+$ field.

Now the equation of motion for incompressible fluid flow is the Navier-Stokes Equation (NSE), which I will restate for convenience,

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t) + f_\alpha(\mathbf{k},t), \qquad (3.12)$$

We take the arbitrary stirring forces $f_\alpha(\mathbf{k},t)$ to satisfy the usual requirements for a well-posed problem. That is, their direct effect is only felt at very small values of the wavenumber. Apart from that, we shall not specify them, as from our point of view, their only importance lies in their maintaining the stationarity of the velocity field against the viscous dissipation of energy.

It should be noted that the stirring forces and the boundary conditions are now excluded from the region of k-space in which assumptions and approximations will be made. Thus, the physics associated with specific features of a problem is separated from the physics of turbulence within the Kolmogorov picture.

Now let us form the evolution equations for the explicit-scale modes ($\mathbf{u}^-$s) and the implicit-scale modes ($\mathbf{u}^+$s). Our first step is to use equation (3.2) to decompose the velocity field on the left-hand side of equation (3.12), which I will write in a more transparent notation as:

$$\begin{aligned}\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] \left(u_\alpha^-(\mathbf{k},t) + u_\alpha^+(\mathbf{k},t)\right) &= M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t) \\ &\quad + f_\alpha(\mathbf{k},t) \qquad (3.13)\end{aligned}$$

Then, we conditionally average both sides of this, to obtain

$$\begin{aligned}\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha^-(\mathbf{k},t) &= \langle M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t)\rangle_0 - \langle L_0 u_\alpha^+(\mathbf{k},t)\rangle_0, \\ &\quad + f_\alpha(\mathbf{k},t) \qquad (3.14)\end{aligned}$$

where

$$L_0 = \left[\frac{\partial}{\partial t} + \nu_0 k^2\right], \qquad (3.15)$$

as the evolution equation for the explicit scales. (Note that (3.14) is obtained by conditional averaging not just splitting the velocity field.) The operator L_0^{-1} is called the propagator in analogy with field theory. The stirring forces are unaffected by the average in the same way as the explicit-scale modes.

Subtraction of (3.14) from the full NSE then yields the corresponding equation for the implicit scales in the form

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha^+(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t) - \langle M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)\rangle_0 + \langle L_0 u_\alpha^+(\mathbf{k},t)\rangle_0. \tag{3.16}$$

Formally, these are the required evolution equations. But, in order to make use of them, we must decompose the non-linear terms as well. In order to do this, we substitute equation (3.2) into the right-hand side of equation (3.14), thus:

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha^-(\mathbf{k},t) = {} & M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\langle u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t)\rangle_0 \\ & + 2M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\langle u_\beta^-(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 \\ & + M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 \\ & - \langle L_0 u_\alpha^+(\mathbf{k},t)\rangle_0 + f_\alpha(\mathbf{k},t),\end{aligned} \tag{3.17}$$

and invoke equation (3.6), to give the key equation:

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha^-(\mathbf{k},t) - {} & \langle M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 \\ & = M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t) \\ & + H_\alpha(\mathbf{k},t) + f_\alpha(\mathbf{k},t),\end{aligned} \tag{3.18}$$

where

$$H_\alpha(\mathbf{k},t) = 2M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\langle u_\beta^-(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 - \langle L_0 u_\alpha^+(\mathbf{k},t)\rangle_0. \tag{3.19}$$

The same procedure applied to equation (3.16) then results in the equation for the implicit scales as

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha^+(\mathbf{k},t) = 2M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\beta^-(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t) + M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j \left[u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t) - \langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0\right] - H_\alpha(\mathbf{k},t). \tag{3.20}$$

Lastly, we may express the correction term $H_\alpha(\mathbf{k},t)$ in terms of the conditionally-averaged mode couplings only, by using equations (3.7)-(3.11), to show that

$$H_\alpha(\mathbf{k},t) = 2M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\beta^-(\mathbf{j},t)\langle\Delta_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 - L_0\langle\Delta_\alpha^+(\mathbf{k},t)\rangle_0, \tag{3.21}$$

which will be small provided the conditional average is a good approximation of the global average (so $\langle\mathbf{u}^+\rangle_0 = 0$). The terms involving $\mathbf{\Delta}^+$ indicate the degree of coupling between modes which as the band is in the inertial/dissipation range means that these correlations ought to be both local in wavenumber space (from Kolmogorov's hypothesis) and small in magnitude (from decorrelation due to dissipation). The error in neglecting terms of order $\mathbf{H}$ can only be made quantitative in the light of a specific choice of $\mathbf{v}^+$. It can be noted now that the term in (3.18) above that will generate the increment to the viscosity is $\langle M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$ and it is this that must be solved to obtain the new NSE in the band $0 \le k \le k_1$ in the same form as the normal NSE.

In order to complete the elimination of the high-k modes from equation (3.18), we need to obtain an explicit expression for $\langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$. Here we face the fundamental problem of relating the conditional average which will depend on the $\mathbf{u}^-$ field to the global turbulent average whose form is known and is independent of $\mathbf{u}^-$ field.

If the decomposition of the $\mathbf{u}^+$ field into correlated and uncorrelated parts, given in (3.7) is applied to this second conditional moment of the subgrid scales, we get

$$\langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 = \langle v_\beta^+(\mathbf{j},t)v_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 + \langle v_\beta^+(\mathbf{j},t)\Delta_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$$

$$+ \langle \Delta_\beta^+(\mathbf{j},t) v_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 + \langle \Delta_\beta^+(\mathbf{j},t)\Delta_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0. \tag{3.22}$$

However, the leading term on the left is homogeneous and thus gives a factor $\delta(\mathbf{k})$. Therefore, it does not contribute to the left-hand side of (3.18) as there is an $M_{\alpha\beta\gamma}(\mathbf{k})$ term in front of the integral. At each order of the moment expansion, the scheme of operation is to find the term which is leading in the $\mathbf{v}^+$'s and compared to which terms of order Δ^+ will be neglected. Consequently, to get a non-trivial contribution to the equation for the explicit scales, (3.18), it will be necessary to look at the full equation for the $\langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$.

We obtain an evolution equation for this quantity from (3.20), and this takes the form

$$\begin{aligned}
&M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j \langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 = \\
&\quad M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j \int_{-\infty}^{t} dt' exp(-\nu_0 j^2 - \nu_0|\mathbf{k}-\mathbf{j}|^2)(t-t') \times \\
&\qquad M_{\beta\delta\epsilon}(\mathbf{j})\int d^3\mathbf{p}[4u_\delta^-(\mathbf{p},t')\langle u_\epsilon^+(\mathbf{j}-\mathbf{p},t')u_\gamma^+(\mathbf{k}-\mathbf{j},t')\rangle_0 \\
&\qquad\quad + 2\langle u_\delta^+(\mathbf{p},t')u_\epsilon^+(\mathbf{j}-\mathbf{p},t')u_\gamma^+(\mathbf{k}-\mathbf{j},t')\rangle_0 \\
&\qquad\quad - 2\langle\langle u_\delta^+(\mathbf{p},t')u_\epsilon^+(\mathbf{j}-\mathbf{p},t')\rangle_0 u_\gamma^+(\mathbf{k}-\mathbf{j},t')\rangle_0] \\
&\qquad\quad - 2\langle u_\gamma^+(\mathbf{k}-\mathbf{j},t')H_\beta(\mathbf{j},t)\rangle_0.
\end{aligned} \tag{3.23}$$

It should be noted that this solution contains the triple conditional moment $\langle \mathbf{u}^+\mathbf{u}^+\mathbf{u}^+\rangle_0$, and that we can solve equation (3.20) for this, in terms of the quadruple conditional moment $\langle \mathbf{u}^+\mathbf{u}^+\mathbf{u}^+\mathbf{u}^+\rangle_0$, and so on. Hence, the hierarchy closure problem is still with us.

Now decomposing the right-hand side of (3.23) using (3.7), and representing certain terms schematically,

$$\begin{aligned}
&M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j \langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 = \\
&\quad M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j \int_{-\infty}^{t} dt' exp(-\nu_0 j^2 - \nu_0|\mathbf{k}-\mathbf{j}|^2)(t-t') \times
\end{aligned}$$

$$
\begin{aligned}
M_{\beta\delta\epsilon}(\mathbf{j}) \int d^3\mathbf{p} [& 4u_\delta^-(\mathbf{p},t')\langle v_\epsilon^+(\mathbf{j}-\mathbf{p},t')v_\gamma^+(\mathbf{k}-\mathbf{j},t')\rangle_0 \\
& + u^-\langle v^+\Delta^+\rangle_0 + u^-\langle v^+\Delta^+\rangle_0 + u^-\langle \Delta^+\Delta^+\rangle_0 \\
& + 2\langle v_\delta^+(\mathbf{p},t')v_\epsilon^+(\mathbf{j}-\mathbf{p},t')v_\gamma^+(\mathbf{k}-\mathbf{j},t')\rangle_0 \\
& + 2\langle \Delta^+ v^+ v^+\rangle_0 + \langle v^+ v^+\Delta^+\rangle_0 + 2\langle \Delta^+ v^+\Delta^+\rangle_0 \\
& + \langle \Delta^+\Delta^+ v^+\rangle_0 + \langle \Delta^+\Delta^+\Delta^+\rangle_0 \\
& + O(\langle \Delta^+\rangle_0). \qquad (3.24)
\end{aligned}
$$

We have the contribution at lowest order of moments in v^+ from the first term on the left-hand side. It is clear that the same problem that was encountered in trying the decomposition in (3.22) above occurs here in dealing with the triple moment. That is, the leading term that represents the uncoupled contribution is identically zero due to the homogeneity of the $\mathbf{v}^+$ moments. Therefore, consistent with what was done previously, the whole term $\langle \mathbf{u}^+\mathbf{u}^+\mathbf{u}^+\rangle_0$ will be treated. The equation for this moment is,

$$
\begin{aligned}
&\langle u_\delta^+(\mathbf{p},t)u_\epsilon^+(\mathbf{j}-\mathbf{p},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 = \\
&\int_{-\infty}^{t} dt' exp[-\nu_0(p^2+|\mathbf{j}-\mathbf{p}|^2+|\mathbf{k}-\mathbf{j}|^2)(t-t')] \times \\
&\quad M_{\delta\lambda\rho}(\mathbf{p}) \int d^3\mathbf{l} [4u_\lambda^-(\mathbf{l},t')\langle u_\rho^+(\mathbf{p}-\mathbf{l},t')u_\epsilon^+(\mathbf{j}-\mathbf{p},t')u_\gamma^+(\mathbf{k}-\mathbf{j},t')\rangle_0 \\
&\qquad + 2\langle u_\gamma^+(\mathbf{l},t')u_\rho^+(\mathbf{p}-\mathbf{l},t')u_\epsilon^+(\mathbf{j}-\mathbf{p},t')u_\gamma^+(\mathbf{k}-\mathbf{j},t')\rangle_0 \\
&\qquad - 2\langle\langle u_\lambda^+(\mathbf{l},t')u_\rho^+(\mathbf{p}-\mathbf{l},t')\rangle_0 u_\epsilon^+(\mathbf{j}-\mathbf{p},t')u_\gamma^+(\mathbf{k}-\mathbf{j},t')\rangle_0] \\
&\quad + M_{\gamma\lambda\rho}(\mathbf{p}) \int d^3\mathbf{l} [2u_\lambda^-(\mathbf{l},t')\langle u_\rho^+(\mathbf{k}-\mathbf{j}-\mathbf{l},t')u_\delta^+(\mathbf{p},t')u_\epsilon^+(\mathbf{j}-\mathbf{p},t')\rangle_0 \\
&\qquad + \langle u_\lambda^+(\mathbf{l},t')u_\rho^+(\mathbf{k}-\mathbf{j}-\mathbf{l},t')u_\epsilon^+(\mathbf{j}-\mathbf{p},t')u_\delta^+(\mathbf{p},t')\rangle_0 \\
&\qquad - \langle\langle u_\lambda^+(\mathbf{l},t')u_\rho^+(\mathbf{k}-\mathbf{j}-\mathbf{l},t')\rangle_0 u_\epsilon^+(\mathbf{j}-\mathbf{p},t')u_\delta^+(\mathbf{p},t')\rangle_0] \\
&\quad + O(\langle \Delta^+\rangle_0). \qquad (3.25)
\end{aligned}
$$

In order to simplify matters, I applied the symmetry conditions that $\delta \leftrightarrow \epsilon$ and $\mathbf{p} \leftrightarrow \mathbf{j}-\mathbf{p}$ which pertain to this third order term as it appears in the hierarchy.

This procedure will continue ad infinitum (or ad nauseam) until the entire hierarchy is revealed for the monster it is. However, the monster can to be claimed to

have been tamed somewhat as we have caged it in a narrow band of wavenumbers located near the dissipation range.

On inspecting the equation for the third order moment one might expect the terms like $\langle \mathbf{u}^+\mathbf{u}^+\rangle_0\langle \mathbf{u}^+\mathbf{u}^+\rangle_0$ to generate terms of order $(\mathbf{u}^-)^2$. However, regarding these terms more closely it can be seen that the contributing terms (i.e. those with only $\mathbf{v}^+$ and $\mathbf{u}^-$ in them) include $\mathbf{u}^-$ terms which have wavenumber arguments that are restricted to the band. These wavenumbers first appear as the arguments of subgrid scale modes and thus carry this restriction implicitly. Thus, the term vanishes as explicit scale modes can only have non-band wavenumber arguments. This argument can be extended to all such terms in higher order terms in the series and so the series is linear in $\mathbf{u}^-$!

The time integrations, which are present in the calculation of the series developed above for the increment to viscosity, will be treated by a Markovian approximation.

3.2.2 The Markovian Approximation

A Markovian approximation involves assuming that a variable only depends for its development on its value at the previous time step. In the case of turbulence theories this implies the neglecting of time histories. I will argue that this is a reasonable assumption based on the time scales of variables approximated. The time scale of a particular mode refers to the time taken for the amplitude at that wavenumber to change appreciably. In some sense, it can be thought of as the time for an eddy or structure of the given length scale to decay, through viscous or inertial interactions.

Rewriting (3.24), considering only the leading term in $\mathbf{u}^-$ on the right-hand side,

we have

$$M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \langle u_\beta^+(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 =$$
$$M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \int_{-\infty}^{t} dt' exp(-\omega_2(j,l)(t-t')) \times$$
$$M_{\beta\delta\epsilon}(\mathbf{j}) \int d^3\mathbf{p} 4 u_\delta^-(\mathbf{p},t') \langle v_\epsilon^+(\mathbf{j}-\mathbf{p},t') v_\gamma^+(\mathbf{k}-\mathbf{j},t')\rangle_0 \tag{3.26}$$

where $\omega_2(j,l) = \nu_0(j^2 + |\mathbf{k}-\mathbf{j}|^2)$, as usual $l = |\mathbf{k}-\mathbf{j}|$.

Using the fact that for the $\mathbf{v}^+$ field the conditional average has the same effect as the normal average and that this has the same properties as the global average of $\mathbf{u}^+$ (i.e. homogeneous, isotropic and stationary), then

$$M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \langle u_\beta^+(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j 4 M_{\beta\delta\epsilon}(\mathbf{j}) D_{\epsilon\gamma}(\mathbf{k}-\mathbf{j}) Q_v^+(|\mathbf{k}-\mathbf{j}|)$$
$$\times \int_0^\infty exp(-\omega_2(j,l)\tau) u_\delta^-(\mathbf{k},t-\tau) d\tau. \tag{3.27}$$

In order to solve this integral, the time dependence of the explicit scales must be known. However, as a principle of this work is to perform the calculation of the subgrid scales before calculating the explicit scales, the effect of this integral must be treated in the context of the separation of the explicit and subgrid scales. If we Taylor expand $u_\delta^-(\mathbf{k},t-\tau)$ around $\tau = 0$ we get

$$\langle u_\beta^+(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 = 4 M_{\beta\delta\epsilon}(\mathbf{j}) D_{\epsilon\gamma}(\mathbf{k}-\mathbf{j}) Q_v^+(|\mathbf{k}-\mathbf{j}|) \frac{u_\delta^-(\mathbf{k},t)}{\omega_2(j,l)}$$
$$+ \text{ higher order time derivatives of } \mathbf{u}^-. \tag{3.28}$$

The higher order terms of (3.28) arise from

$$\int_0^\infty exp(-\omega_2(j,l)\tau) u_\delta^-(\mathbf{k},t-\tau) d\tau$$
$$= \int_0^\infty exp(-\omega_2(j,l)\tau) \left[u_\delta^-(\mathbf{k},t) - \tau \frac{\partial u_\delta^-(\mathbf{k},t)}{\partial t} + O(\tau^2) \right] d\tau \tag{3.29}$$

and thus will involve the decay rate of explicit scale modes and $\omega_2(j,l)$ (decay of two subgrid modes). Hence, the assumption that we can set the time argument of

the $\mathbf{u}^-$ to t is equivalent to taking the zeroth order term in a Taylor series for $\mathbf{u}^-$ about t. It is clear that we must require that the explicit modes be slowly varying in time compared to the subgrid modes.

The higher order terms in the moment hierarchy will have appropriately higher numbers of time integrations which are dealt with in the same way as the above. Each integral with yield another factor of the form $\frac{1}{\omega_n}$ where n is the order of the moment being replaced in the series. In the above it was the second order moment that was replaced, hence the integral gave a $\frac{1}{\omega_2(j,l)}$ factor.

Now it remains to identify a plausible measure of the decay rate of the explicit scale-modes in order to gauge the validity of the neglect of the higher order derivatives compared to $\omega_2(j,l)$. We require the average time scale for decay of an explicit mode of wavenumber k, in the energy-containing range, on dimensional grounds it is expected to be

$$(\omega^-(k))^{-1} = u^-_{rms}k \tag{3.30}$$

where

$$u^-_{rms} = \sqrt{\frac{2}{3}\int_0^{k_1} E(k)dk} \tag{3.31}$$

and consequently the ratio we are interested in is

$$\frac{\omega^-(k)}{\omega_2(j,l)} = \frac{\left(k\sqrt{\frac{2}{3}\int_0^{k_1} E(k)dk}\right)^{-1}}{\nu_0(j^2+|\mathbf{k}-\mathbf{j}|^2)}. \tag{3.32}$$

The explicit scales will have time scales of the energy-containing modes, that is $\tau_L = O(u^{-1}_{rms}k_L^{-1})$, where k_L is a wavenumber in the energy containing modes. The subgrid modes lie in the inertial/dissipation range and their time scale $\tau_d(k)$ is approximately given by $(\nu_0 k^2)^{-1}$. Note that this discussion is in the context of the first-shell elimination and the distinctions made in separating scales into energy-containing and universal ranges only really apply after the fixed point is reached.

When k is in the energy-containing range then it is to be expected that it has a much larger time scale than those modes in the inertial or dissipation ranges. However such a clear division of the behaviour of the explicit and subgrid scales, though desirable, is not obviously justifiable. Consider the behaviour of $\mathbf{u}^-$ at the cut, where the modes are dominated by dissipation, and we see that the decay rates of the $\mathbf{u}^-$ and the $\mathbf{u}^+$ modes near the cut ought to be the same. In this worst possible case, $k = k_1 = j = |\mathbf{k} - \mathbf{j}|$ and so in order to further understand the nature of this approximation, we make a simplifying assumption that the behaviour in time of this $\mathbf{u}^-$ mode may be written,

$$u_\alpha^-(\mathbf{k}_1, t) = u_\alpha^-(\mathbf{k}_1) e^{-\omega^-(k_1)t} \tag{3.33}$$

then the Taylor series for $\mathbf{u}^-$ gives a series of integrals which sum as a geometric series,thus

$$\begin{aligned}
&\int_0^\infty exp(-\omega_2(j,l)\tau) u_\delta^-(\mathbf{k}_1, t-\tau) d\tau \\
&\quad = \int_0^\infty exp(-\omega_2(j,l)\tau) u_\delta^-(\mathbf{k}_1) e^{-\omega^-(k_1)t} \left[1 + \tau\omega^-(k_1) + \frac{(\tau\omega^-(k_1))^2}{2} + O(\tau^3)\right] d\tau \\
&\quad = \frac{u_\delta^-(\mathbf{k}_1, t)}{\omega_2(j,l) - \omega^-(k_1)}.
\end{aligned} \tag{3.34}$$

We have used

$$\int_0^\infty x^n e^{-\alpha x} dx = n! \alpha^{-n-1}. \tag{3.35}$$

At the cut,

$$\omega^-(k_1) = \nu(k_1) k_1^2 \tag{3.36}$$

hence when $k = k_1 = j = |\mathbf{k} - \mathbf{j}|$ the zeroth order is out by 50%. However, it has to be stressed that the whole band is integrated over and when j is near the upper cut, k_0, this error will be reduced.

As has been stated, the above discussion is in the context of the first shell iteration (i.e. the elimination of the first shell of wavenumbers). As the iteration reaches a fixed point it can be seen that a genuine separation of behaviours occurs. The

explicit scales will be the energy-containing scales while the subgrid scales will be those with universal characteristics. In this context, there are good grounds to believe that as the effect of the inertial range scales is subsumed by the effective viscosity then the above picture of an exponential time-decay in the band will be correct and that the time scale of the explicit modes ought to be decoupled and much greater than that of the subgrid modes.

If the time scale of the energy-containing modes,$\tau_L(k)$, is dynamically significant then the inertial range would have a $k^{-3/2}$ spectrum. This is shown in the Eulerian formulation of Kraichnan's theory, the Direct Interaction Approximation [39], where the approximations lead to the fact that sweeping by the energy-containing eddies is vital to energy transport in the inertial range. Experiment has shown [6] that the Kolmogorov spectral form is the correct form and thus it is widely held that these time scales ought not to matter [40].

Note, there is no need to consider separately the time behaviour of the $\mathbf{v}^+$ field as its moments are stationary.

3.2.3 The Moment Hierarchy

We now know that the series of moments is linear in $\mathbf{u}^-$ and have established the correct form of the first term in the series. The next task is to calculate the sum of the series in order to obtain the increment to viscosity. It can be seen that each successive term of the series will involve a higher order moment of the $\mathbf{v}^+$s, consequently their distribution must be known in order to proceed. It is assumed that that distribution is Gaussian which in the light of the later choice of $\mathbf{v}^+$ in (3.44) or (3.45) seems plausible as $\mathbf{u}^+(\mathbf{k}_0)$ is heavily damped by viscosity and thus ought to have nearly the expected, Gaussian, distribution for the viscous range modes. This choice for the statistics greatly simplifies the task of writing the

series as a power series since all odd order moments vanish and the even order moments break down into products of Q_v^+ factors. However, the general term of the series is still highly complex, involving as it does many convolution sums. It appears impracticable to attempt to work out the general term exactly and thus some more qualitative features of the series are investigated.

Firstly, I tried to find an upper bound for the sum. This involves considering a series where each term is bigger than the corresponding term in the $\delta\nu(k)k^2$ series. The details of this work are relegated to Appendix A. The series thus obtained is divergent(i.e. its sum is infinite). This is not the disaster it might first appear as it is common for problems in physics to yield divergent perturbation series for finite problems. It implies that there is a problem in the formulation of the series (e.g. incorrect choice of expansion parameter). Here, there is no clear expansion parameter as the series is in increasing order of moments of $\mathbf{v}^+$. The bounding series is Poincare asymptotic and hence there is the possibility of a reordering of the series yielding a sum. As the cause of the divergence was a factorial coefficient in the general term, a Borel resummation was attempted. A Borel sum is a reordering of the series in integral form which can give a finite value for the sum in certain limits.

The Borel sum turns out to be an improper integral (i.e. non-finite) but it must be remembered that the bounding series is a very crude approximation to the actual series for $\delta\nu(k)k^2$. By relaxing the condition that we must find a strict upper bound for the sum and modelling the series in an obvious way (details in Appendix A), an approximate sum for the series was obtained. The value of this is it shows that as we shrink the band width of wavenumbers being averaged out so the first term in the series will represent a good approximation to the sum of the series.

The equation for the explicit modes can be rewritten as

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha^-(\mathbf{k},t) - T_{\alpha\beta}(\mathbf{k})u_\beta^-(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t) + f_\alpha(\mathbf{k},t), \tag{3.37}$$

where for isotropic fields, we may further write

$$T_{\alpha\beta}(\mathbf{k})u_\beta^-(\mathbf{k},t) = \frac{1}{2}T_{\beta\beta}(\mathbf{k})u_\alpha^-(\mathbf{k},t) \tag{3.38}$$

(see Appendix B). Evidently this is a form which justifies our representing the effect of the coupling to the implicit scales in terms of an increment to the viscosity acting on the explicit scales. That is, we rewrite equation (4.20) in the form

$$\left[\frac{\partial}{\partial t}+(\nu_0+\delta\nu_0)k^2\right] u_\alpha^-(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t) + f_\alpha(\mathbf{k},t), \tag{3.39}$$

With the procedures discussed above, we can derive an expression for the increment $\delta\nu_0$ as the first term in the series generated by the moment hierarchy. The essential point is that this elimination of the first shell of modes $(1-\lambda)k_0 < k < k_0$ is a quite general result. However, in order to carry out the second step which completes one iteration of the renormalisation group, we must find the second moment of the $\mathbf{v}^+$ field. In doing this, the effect of the approximations already made should appear quantifiable.

3.2.4 The first shell elimination

Then, with all these points in mind, equation (3.39) yields for the viscosity acting on the explicit scales:

$$\nu_1 = \nu_0 + \delta\nu_0, \tag{3.40}$$

where the formula for the increment to viscosity is

$$\delta\nu_0(k) = \frac{1}{k^2}\int d^3j \frac{L(\mathbf{k},\mathbf{j})Q_v^+(|\mathbf{k}-\mathbf{j}|)}{\nu_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2}, \tag{3.41}$$

with

$$0 \leq k \leq k_1; k_1 \leq j, |\mathbf{k}-\mathbf{j}| \leq k_0,$$

where the coefficient $L(\mathbf{k},\mathbf{j})$ is given by

$$\begin{aligned} L(\mathbf{k},\mathbf{j}) &= -2M_{\rho\beta\gamma}(\mathbf{k})M_{\beta\rho\delta}(\mathbf{j})D_{\delta\gamma}(|\mathbf{k}-\mathbf{j}|) \\ &= -\frac{[\mu(k^2+j^2)-kj(1+2\mu^2)](1-\mu^2)kj}{k^2+j^2-2kj\mu}, \end{aligned} \tag{3.42}$$

where μ is the cosine of the angle between the vectors $\mathbf{k}$ and $\mathbf{j}$.

Now let us consider our principal approximation. We begin by noting that the choice of the $\mathbf{v}^+$ field is arbitrary, apart from the constraints embodied in equations (3.8), (3.9) and (3.11). But we cannot expect thereby, that our results will be insensitive to the nature of this choice. Indeed, we must expect that a choice which best reflects the underlying physics will give optimum results, if only because this determines the accuracy of the approximation of neglecting terms involving $\mathbf{\Delta}^+$.

Consider the Taylor series expansion for $u_\alpha^+(\mathbf{k},t)$ about $\mathbf{k}=\mathbf{k_0}$,

$$u_\alpha^+(\mathbf{k},t) = u_\alpha^+(\mathbf{k_0},t) + (\mathbf{k}-\mathbf{k_0}).\nabla u_\alpha^+(\mathbf{j},t)|_{j=k_0} + \text{higher order terms.} \tag{3.43}$$

It is clear that a representation of $\mathbf{v}^+$ by a truncation of this series will face two constraints. Firstly, we require that $\mathbf{v}^+$ be independent of $\mathbf{u}^-$. Therefore, λ must be large enough for $\mathbf{u}(\mathbf{k_0})$ to be independent of $\mathbf{u}(\mathbf{k_1})$. This form for $\mathbf{v}^+$ then approximately satisfies the Kolmogorov assumption of locality of interaction in k-space. Secondly, λ must be small enough for us to represent the Fourier components in the band by means of this truncated Taylor series, without the remainder terms, the $\mathbf{\Delta}^+$s becoming significant. In this way, we impose both upper and lower bounds on λ, when we make the identifications

$$v_\alpha^+(\mathbf{k},t) = u_\alpha^+(\mathbf{k_0},t)\text{- the zeroth order approximation,} \tag{3.44}$$

or

$$v_\alpha^+(\mathbf{k},t) = u_\alpha^+(\mathbf{k_0},t) + (\mathbf{k}-\mathbf{k_0}).\nabla u_\alpha^+(\mathbf{k},t)|_{k=k_0}\text{- the first order approximation.} \tag{3.45}$$

Note that I am using λ where properly I should refer to $(\mathbf{k}-\mathbf{k_0})$ as the expansion parameter. However, the maximum value of $(\mathbf{k}-\mathbf{k_0})$ is λk_0, since $\mathbf{k}$ is restricted to the band $k_1 \leq k \leq k_0$ and $\mathbf{k_0}$ is chosen parallel to $\mathbf{k}$. Therefore, this series is bounded by one in λ, which is a more convenient parameter with which to discuss the theory.

We may evaluate Q_v^+ , using equation (3.45) for the first order approximation, with the result that the increment to the viscosity takes the form

$$\delta\nu_0(k) = \frac{1}{k^2}\int d^3j \frac{L(\mathbf{k},\mathbf{j})\{Q(l)|_{l=k_0} + (l-k_0)\frac{\partial Q(l)}{\partial l}|_{l=k_0}\}}{\nu_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2}, \tag{3.46}$$

where $l = |\mathbf{k}-\mathbf{j}|$. Hence, we have succeeded in getting the increment into the form we need to be able to perform the renormalisation transformation on the Navier-Stokes equation and that completes the elimination of the first shell of modes. We extend the procedure to further shells, as follows:

(a) Set $u_\alpha^-(\mathbf{k},t) = u_\alpha(\mathbf{k},t)$ in equation (3.39), so that we now have a new NSE with effective viscosity $\nu_1(k)$ for Fourier modes on the interval $0 < k < k_1$.

(b) Make the decomposition of (3.2), but this time at $k = k_2$, such that $u_\alpha^+(\mathbf{k},t)$ is now defined in the band $k_2 \leq k \leq k_1$.

(c) Repeat the procedures used to eliminate the first shell of modes in order now to eliminate modes in the band $k_2 \leq k \leq k_1$.

In this way, we can progressively eliminate the effect of high wavenumbers in a series of bands $k_{n+1} < k < k_n$, where

$$k_n = (1-\lambda)^n k_0 \; ; \; 0 \leq \lambda \leq 1, \tag{3.47}$$

with, by induction, the recursion relation for the effective viscosity given by

$$\nu_{n+1}(k) = \nu_n(k) + \delta\nu_n(k), \tag{3.48}$$

where the increment of order n takes the form

$$\delta\nu_n(k) = \frac{1}{k^2}\int d^3j \frac{L(\mathbf{k},\mathbf{j})\{Q(l)|_{l=k_n} + (l-k_n)\frac{\partial Q(l)}{\partial l}|_{l=k_n}\}}{\nu_n(j)j^2 + \nu_n(|\mathbf{k}-\mathbf{j}|)|\mathbf{k}-\mathbf{j}|^2}, \tag{3.49}$$

At the same time, the evolution equation for the explicit scales may be written (inductively, from equation (3.39)), as

$$\left[\frac{\partial}{\partial t} + \nu_n(k)k^2\right] u_\alpha^-(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t) + f_\alpha(\mathbf{k},t), \tag{3.50}$$

for wavenumbers

$$0 \le k \le k_{n+1}; k_{n+1} \le j, |\mathbf{k}-\mathbf{j}| \le k_n.$$

3.2.5 The Scaled Equations for Effective Viscosity

An energy equation for the explicit scales can now be formed. We do this by multiplying each term of equation (3.50) by $u_\alpha^-(\mathbf{k},t)$ and averaging. If we then integrate each term in the resulting balance equation with respect to wavenumber over the range $0 < k < k_n$, we obtain the renormalised dissipation relation, viz:

$$\int_0^{k_n} 2\nu_n(k)k^2 E(k)dk = \epsilon, \tag{3.51}$$

which may be compared with the unrenormalised form given in equation (3.1).

If we now assume that the energy spectrum in the band is given by a power law of the form:

$$E(k) = \alpha\epsilon^r k^s$$

and make the scaling transformation

$$k_{n+1} = hk_n, \tag{3.52}$$

where, for compactness, we define $h = (1-\lambda)$. The effective viscosity may be written, from (3.48) and (3.49), as

$$\nu(k_n k') = \alpha^{1/2}\epsilon^{r/2}k_n^{(s-1)/2}\tilde{\nu}_n(k'). \tag{3.53}$$

Substitution of (3.53) into (3.51) fixes the exponents as $r = 2/3$ and $s = -5/3$: the well known Kolmogorov spectrum. With these results, I can scale equations (3.53), (3.48) and (3.49). Firstly, the scaled effective viscosity is

$$\nu_n(k_n k') = \alpha^{1/2}\epsilon^{1/3}k_n^{-4/3}\tilde{\nu}_n(k'). \tag{3.54}$$

It is now possible to scale the recursion relation and the increment to viscosity, details of which appear in Appendix C,

$$\tilde{\nu}_{n+1}(k') = h^{4/3}\tilde{\nu}_n(hk') + h^{-4/3}\delta\tilde{\nu}_n(k'). \tag{3.55}$$

In the equation above the expression for the increment to viscosity has been scaled by the upper wavenumber in the (n+1)th cycle, k_{n+1}, to

$$\delta\tilde{\nu}_n(k') = \frac{1}{4\pi k'^2}\int d^3j' \frac{L(\mathbf{k}',\mathbf{j}')Q'}{\tilde{\nu}_n(hj')j'^2 + \tilde{\nu}_n(hl')l'^2} \tag{3.56}$$

for the wavenumber bands

$$0 \leq k' \leq 1; 1 \leq j', l' \leq h^{-1}$$

where $l' = |\mathbf{k}' - \mathbf{j}'|$ and

$$Q' = h^{11/3} - \frac{11}{3}h^{14/3}(l' - h^{-1}) + h.o.t. \tag{3.57}$$

The form given above for Q_v^+ is for the first order approximation; the zeroth order form is easily obtained by taking only the first term on the right-hand side of (3.57). It can be seen that although the range of the integration for the increment lies outside the normal range for scaled wavenumbers(i.e. 0 to 1), the evaluations of scaled effective viscosities are carried out within the range of argument for which they exist.

Iteration of equations (3.55) and (3.56) reaches a fixed point, with

$$\tilde{\nu}_{n+1}(k') = \tilde{\nu}_n(k') = \nu^*(k'). \tag{3.58}$$

The results of the calculations of this quantity and the Kolmogorov constant for the zeroth and first order approximations are given in the next chapter.

3.3 Comments

Viewing the above work in the light of the Kolmogorov picture of turbulence, it can be seen that at a number of points, the principle of local interaction of modes is used. It is explicitly assumed that the energy spectrum in the band is the power law form that leads to the famous '-5/3' law. Assuming that in the inertial range only modes of similar wavenumber strongly interact produces the necessary separation of behaviour on different scales. It should be noted that all other results are consistent with this.

The Markovian approximation for the inverse propagators that appear in the moment hierarchy can also be viewed as arising out of the locality of the interactions where only the time scales appropriate to the band are relevant not those of the explicit(non-universal) modes. However, this is not exact as, if the band is too narrow then modes on either side of the cut, between explicit and subgrid scales, will have the same time scales. Hence, I would expect (as is the case), that the theory would break down if λ is too small. A further discussion of the sensitivity of the theory to its assumptions will be given in the next chapter.

The velocity field is split twice, the first occasion is the usual LES decomposition into large scale and subgrid scale modes. This can be seen as dividing the region of system-specific physics from that of the universal physics of turbulence, although this description only really becomes appropriate after the fixed point is reached. The second decomposition is that of the subgrid modes into the v^+ and Δ^+ modes. Again one may look at this as an attempt to find the variables which will give the universal behaviour as opposed to the Δ^+ modes which will be dependent on the explicit scales for their statistical distribution. One may now see the logic of introducing a conditional average over the subgrid modes which will therefore only act on that region of k-space which contains the universal behaviour.

In order to quantify further the nature of the neglect of the effect of Δ^+ modes, the relation of these modes to the explicit scale modes needs to be investigated. This would resurrect the prospect of dealing with an infinite series in the full velocity field and not the restricted one I have created here. It is worth noting again that the derivation here avoids a power series in the explicit modes, this being contained in the Δ^+ terms, and delivers the increment term as expected as a function of $|\mathbf{k}|$, only.

Another way of considering the neglect of the H and Δ^+ terms, is to realise that if the u^+ modes were homogeneous under the conditional average then all H terms and the hierarchy generating term ($\langle u^+u^+u^+\rangle_0$) vanish exactly. This would give a form for the increment to viscosity of the form above but with Q_u^+ instead of Q_v^+. We can picture this as the explicit modes acting as a decoupled background in which the subgrid modes merely evolve according to the NSE(obtaining the appropriate distribution). Obviously, this cannot be the case as it would involve a discontinuous jump in the dynamics of the mode picture. However, at some fixed point, where I would claim that the universal physics of the inertial range dominate, a field approximating the subgrid field will behave thus and I would claim that the obvious choice is that of the Taylor series expansion about the upper cut-off.

In their recent paper, Domaradzki and Rogallo[41] have studied results from direct numerical simulations at moderate Reynolds' numbers to examine the effect of different triad interactions on energy transfer. A wavenumber triad is the group of three wavenumber arguments of velocity field modes that appear as the transfer term in the energy equation ($T(k,t)$ in equation (1.20)). This work concludes that energy transfer is dominated by interactions between modes where one mode is large scale and the others are smaller and approximately equal. These particular triads of wavenumbers are found to be more effective than those with all three legs of the triad roughly equal. The equivalent equation in this work is (3.24)

where these results support the argument that the hierarchy can be approximated by the leading term which represents the $\langle u^- v^+ v^+ \rangle_0$ contribution as against the triple moment term, which generates the rest of the hierarchy.

The form of the increment to viscosity is similar to that obtained from DIA by assuming exponential time dependences for the propagator and the correlation function. There are two differences, firstly, that instead of the correlation function for u I have a correlation for the v^+ field. Secondly, the integral over the dummy wavenumber argument is restricted in (3.41) to the band of subgrid wavenumbers. This is crucial in enabling the approximations made in the derivation to be justified.

Chapter 4

Results and Analysis of the Velocity Field Problem

4.1 Introduction

In this chapter, I will firstly show how the theory derived in chapter 3 may be used to form an expression for the Kolmogorov constant, α, and then outline the computation of the iteration to a fixed point in the effective viscosity and the numerical calculation of α.

Secondly, I will give the results of these calculations in some detail in order to examine the assumptions and approximations made in the theory. The existence of the fixed point for the effective viscosity will be demonstrated and the stability of the iteration tested. I will discuss the results in terms of other calculations for the effective viscosity and the Kolmogorov constant. A comparison will be drawn with the experimental results that exist.

Finally, I will discuss how the theory and results may be applied to a large-eddy simulation.

4.2 An Integral Expression for the Kolmogorov Constant

In the last chapter, I derived an expression for the effective viscosity which must be iterated under a certain recursion relation to obtain the fixed point value. Once a fixed point is found we can calculate the Kolmogorov constant from the expression

$$\epsilon = \int_0^{k_0} 2\nu_0 k^2 E(k) dk. \tag{4.1}$$

If this is expressed for the situation after the fixed point has been reached at, say, the Nth cycle then

$$\epsilon = \int_0^{k_N} 2\nu_N(k) k^2 E(k) dk \tag{4.2}$$

where E(k) is the Kolmogorov energy spectrum. Using the Kolmogorov spectrum and the expression for the scaled viscosity,

$$E(k) = \alpha \epsilon^{2/3} k^{-5/3} \tag{4.3}$$

and

$$\nu_n(k) = \alpha^{1/2} \epsilon^{1/3} k_n^{-4/3} \tilde{\nu}_n(k') \tag{4.4}$$

we obtain the desired equation for the Kolmogorov constant,

$$\alpha = \left\{ 2 \int_0^1 \nu^*(k') k'^{1/3} dk' \right\}^{-2/3}, \tag{4.5}$$

as first given by McComb[29].

This result is only valid in the limit of $Re \to \infty$ since it has been assumed that the Kolmogorov spectrum is true for all wavenumbers. This can be pictured using the thought experiment of Edwards[24] where turbulence is stirred by a delta function at $k = 0$ and as $Re \to \infty$ will dissipate via a delta function at $k = \infty$. Thus the formula should be formally altered to

$$\alpha = \left\{ 2 \int_{0+}^1 \nu^*(k') k'^{1/3} dk' \right\}^{-2/3}. \tag{4.6}$$

4.3 Calculation of Results

The calculation of the fixed point values for the effective viscosity and thus the Kolmogorov constant were carried out on the VAX system of the Edinburgh University Computing Service. The code was written in FORTRAN and is based on work done by Dr. V. Shanmugasundaram in earlier calculations of the Iterative Averaging theory[29].

The calculation of the fixed point was not done using $\nu_n(k)$ but instead $\omega_n(k) = \nu_n(k)k^2$. This cancels the difficult factor of $\frac{1}{k^2}$ in equation (3.49) which causes large rounding errors in the low k' range of the integration. It avoids the introduction of an asymptotic formula for the integral which was used previously.

Therefore, it is necessary to derive the increment formula in terms of $\tilde{\omega}_n(k')$ and the appropriate recursion relation. The scaling relation for $\omega_n(k)$ is

$$\omega_n(k_n k') = \alpha^{1/2}\epsilon^{1/3}k_n^{2/3}\tilde{\omega}_n(k'). \tag{4.7}$$

Thus the recursion relation can be derived from that for the effective viscosity as

$$\tilde{\omega}_{n+1}(k') = h^{-2/3}\tilde{\omega}_n(hk') + h^{2/3}\delta\tilde{\omega}_n(k'). \tag{4.8}$$

The increment to $\omega_n(k)$ is therefore,

$$\delta\omega_n(k) = \int d^3j \frac{L(\mathbf{k},\mathbf{j})\{Q(l)|_{l=k_n} + (l-k_n)\frac{\partial Q(l)}{\partial l}|_{l=k_n}\}}{\omega_n(j) + \omega_n(|\mathbf{k}-\mathbf{j}|)}, \tag{4.9}$$

which when scaled gives

$$\delta\tilde{\omega}_n(k') = \frac{1}{4\pi}\int d^3j' \frac{L(\mathbf{k}',\mathbf{j}')Q'}{\tilde{\omega}_n(hj') + \tilde{\omega}_n(hl')} \tag{4.10}$$

with the wavenumbers confined in the usual way,

$$0 \le k' \le 1; 1 \le j', l' \le h^{-1}$$

where $l' = |\mathbf{k}' - \mathbf{j}'|$.

I will give a simplified listing of the code to outline the method.

1. Input parameters from a data file
2. Discretise the k' range
3. For all λ values
   a) Discretise the j',μ range
   b) Calculate the increment to $\tilde{\omega}_n(k')$, equation (4.10), by quadrature
   c) Calculate $\tilde{\omega}_{n+1}(k')$ from the recursion relation, equation (4.8)
   d) Check for convergence of $\tilde{\omega}_n(k')$ to a fixed point.
      If unconverged then do another iteration cycle, go back to 3.b.
      However, if converged then
   e) Output all effective viscosity data
   f) Use the value of $\omega^*(k')$, the fixed point value, to calculate the Kolmogorov constant, α. (Numerical integration by quadrature of equation (4.6))
   g) Output the value of α
4. Select the next value of λ, go back to 3.a

The chosen test for convergence is to check each value of $\tilde{\omega}_n(k')$ at each k' array point and only accept convergence to the fixed point when all values of $\tilde{\omega}_{n+1}(k')$ are within a tolerance of 0.1% of $\tilde{\omega}_n(k')$ values.

4.4 Presentation and Discussion of the Results

These results are generated by the code above with alterations for different forms of the increment to viscosity. Results for all values of λ for which convergence has been found are given.

The main results are for the zeroth and first order approximations given in Chapter 3. Detailed results are produced for these approximations in three regions of the λ range. These correspond to the plateau region of the α v. λ plot, where the behaviour of the effective viscosity is independent of the bandwidth; the high λ region where the breakdown of the Taylor series is observed; and the low λ region where the approximation for v^+ by the Taylor series about $\mathbf{u}(\mathbf{k_0})$ breaks down due to the band being too narrow and thus the failure of $\mathbf{u}(\mathbf{k_0})$(the approximation to the v^+ field) to be independent from the explicit scales. I have also included, for comparison, some results of calculations using an approximation to a second order in the Taylor series and using the full Kolmogorov spectral form for Q_v^+.

If the Taylor series were taken to the second order then one gets terms which can be formed into the second derivative of the energy spectrum and also terms which are the correlations of the first derivatives of the velocity field at the cut. These derivative terms cannot be treated simply. Consequently, I have neglected them and used the form of the energy spectrum taken to the second order in a Taylor series about the upper cut-off, as follows

$$Q_v^+(k) = Q(k_0) + (k-k_0).\frac{\partial Q(l)}{\partial l}|_{l=k_0} + \frac{(k-k_0)^2}{2}.\frac{\partial^2 Q(l)}{\partial l^2}|_{l=k_0} \qquad (4.11)$$

and hence the scaled increment formula is as (4.10) with Q' as

$$Q' = h^{+11/3} - \frac{11}{3}(l'-h^{-1})h^{+14/3} + \frac{77}{9}(l'-h^{-1})^2 h^{+17/3}, \qquad (4.12)$$

where as before, $l' = |\mathbf{k}' - \mathbf{j}'|$.

The assumption of the full Kolmogorov energy spectrum is equivalent to assuming that $v^+ = u^+$ and hence,

$$\begin{aligned} Q_v^+(k) &= Q(k) \\ &= \frac{\alpha}{4\pi}\epsilon^{2/3}k^{-11/3}. \end{aligned} \tag{4.13}$$

The scaled increment is as (4.10) with

$$Q' = l'^{-11/3}. \tag{4.14}$$

<u>α v. λ</u>

In figure 4.1, the three regions identified above can be seen for the zeroth and first order approximations. There is a region on both curves where the value of α is independent of the bandwidth. Obviously, in the spirit of RG, physical quantities ought to be independent of the averaging procedure used to derive them. Therefore, ideally this plateau regime ought to extend over all λ values. However, the theory is approximate and as stated above there are good reasons for its failure at either end of the bandwidth range. The first order shows a range of λ values for which the Kolmogorov constant is independent of λ much larger than that for the zeroth order where the breakdown of the Taylor series effects the results at much smaller values of λ. The graphs show a tendency to converge as $\lambda \to 0$. A precise convergence would be expected if the Taylor series were the only approximation, however, the breakdown in the use of $\mathbf{u}(\mathbf{k_0})$ as independent of the explicit scales effects the results in this region.

Figure 4.2 displays the same graphs with those for the second order approximation and the Kolmogorov spectrum. The Kolmogorov spectrum ought to represent the case where the approximation for the v^+ field is poorest in obtaining independence from the explicit scales and it is notable that the zeroth and first order approximations behave in the same way in the small λ region. This supports the above argument for the breakdown in this region. However, the graph also exhibits λ

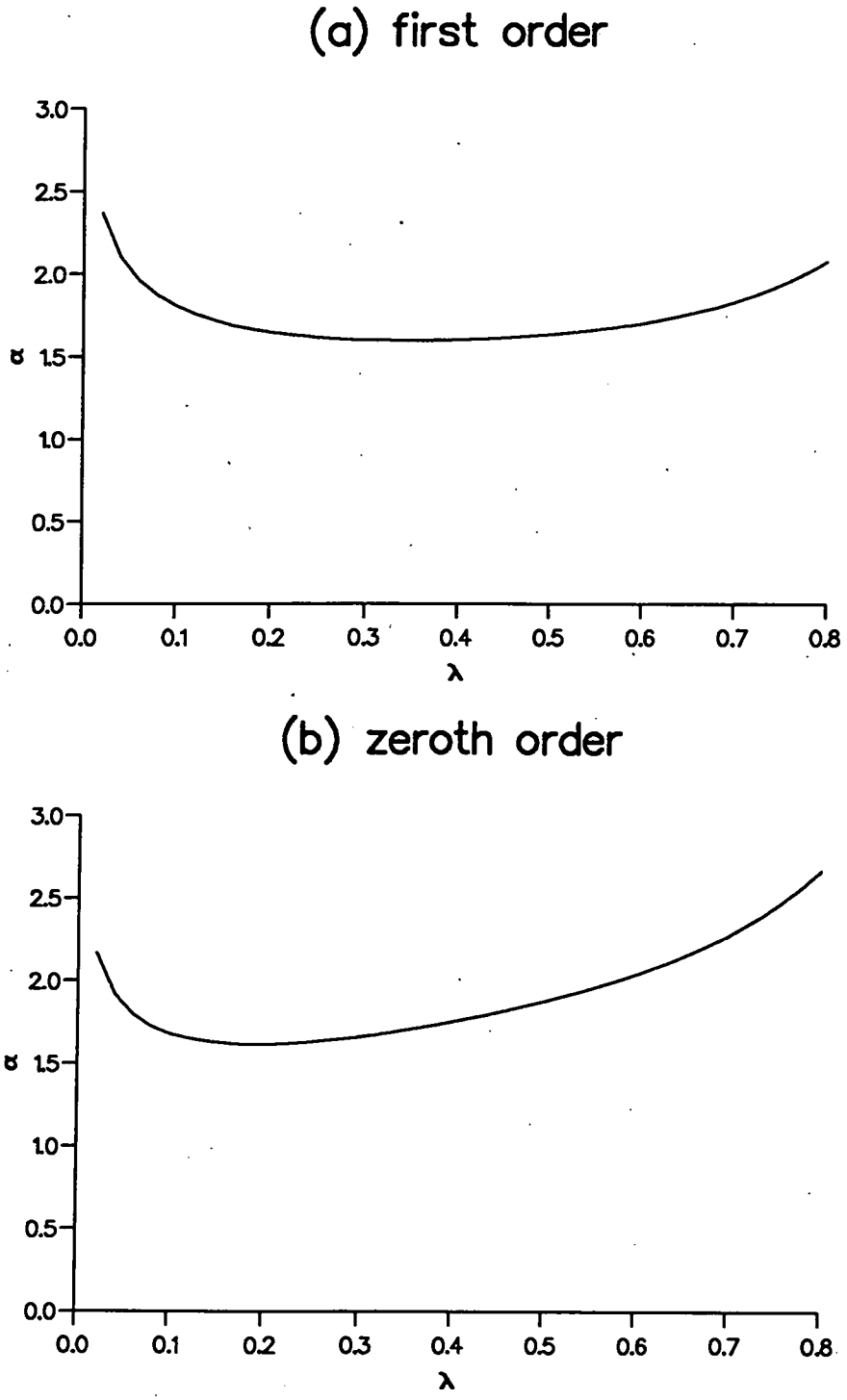

Figure 4.1: Kolmogorov constant versus bandwidth parameter

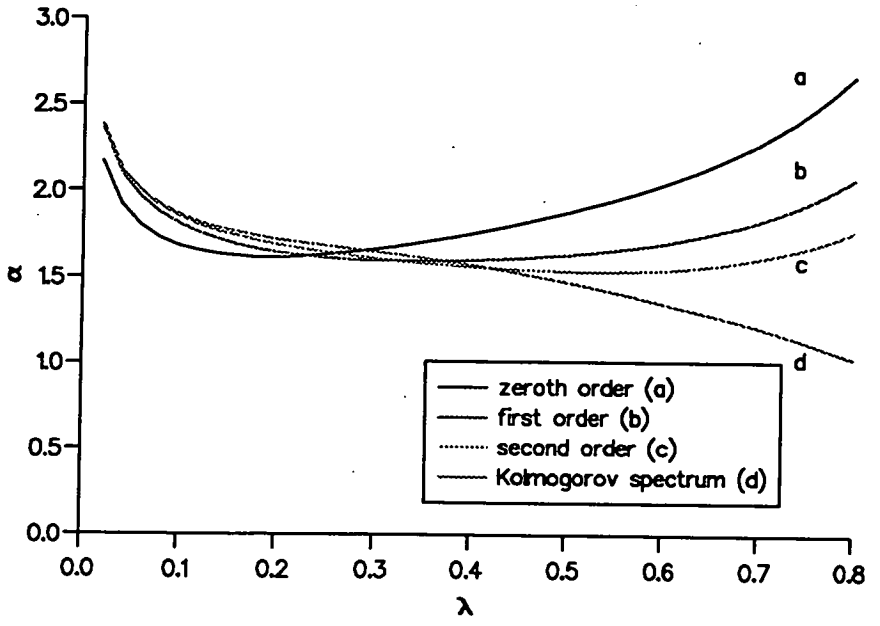

Figure 4.2: Kolmogorov constant versus bandwidth parameter

dependence in the large λ region indicating that a simple continuation of the Taylor series to all orders in λ (which the Kolmogorov spectrum results represent) does not work. This illustrates the fundamental importance of need to separate the explicit and subgrid scales.

The arguments above lead to the selection of the value of the Kolmogorov constant from the plateau region. The first order yields a value of 1.60 ± 0.01 in range $\lambda = 0.25 - 0.45$. The zeroth order yields a result of 1.62 ± 0.01 in the range $\lambda = 0.14 - 0.25$. These results lie in the generally accepted range of values for α which according to Monin and Yaglom[9] is 1.2-1.8. A widely quoted value for α is that obtained from Grant, Stewart and Moilliet[6] at 1.44. However, it has been noted by Leslie[44] that Kraichnan[45] has pointed out that there are difficulties in fitting a curve to the experimental data for the energy spectrum and Leslie suggests that had a better fitting procedure been used then a value of between 1.6 and 1.7 would have been obtained. Clearly, the value of α does not represent

a definitive test of a theory. Although, it seems fair to conclude that the theory above is in good agreement with the theoretical and experimental values for α.

The α value is not exactly independent of λ, as the graph exhibits more of a minimum than an absolutely flat plateau. However, the results of the α calculation depend on the fixed point of the effective viscosity, which is only calculated to 0.1%. The error given for the α value reflects this although calculations at higher precision indicate that this error is pessimistic. Thus, it may well be that there is a minimum for this graph. However, it is justified in calling this region a plateau to an accuracy of about 1%.

Further discussion on the shape of these plots is given when the results for $\nu^*(k')$ and its variation with λ are discussed with respect to figure 4.7.

$\nu^*(k')$ v. k'

In figure 4.3, the general form of the effective viscosity can be observed. The most obvious comment that can be made is that $\nu^*(k')$ shows a clear k' dependence. The function decreases as the supergrid-subgrid cut is reached unlike the effective viscosities of Kraichnan's Test Field Model[35] and direct numerical simulation results [37]. However, as has been pointed out Kraichnan's cusp is due to an integrable singularity at the origin when the Kolmogorov spectrum is assumed to be valid for the whole wavenumber range[3] and Domaradzki et al.[37] produce results for very low Reynolds' numbers which preclude the existence of an inertial range which the effective viscosity, presented here, is tackling.

The shape of the effective viscosity is very similar to that of Leslie and Quarini[38] when the supergrid scales only include those in the energy-containing range of wavenumbers. The cusp behaviour (see Chapter 2) is only apparent when the LES includes inertial range wavenumbers as well. The theory described here yields an effective viscosity which accounts for the inertial range and thus leaves only the energy-containing range to be simulated.

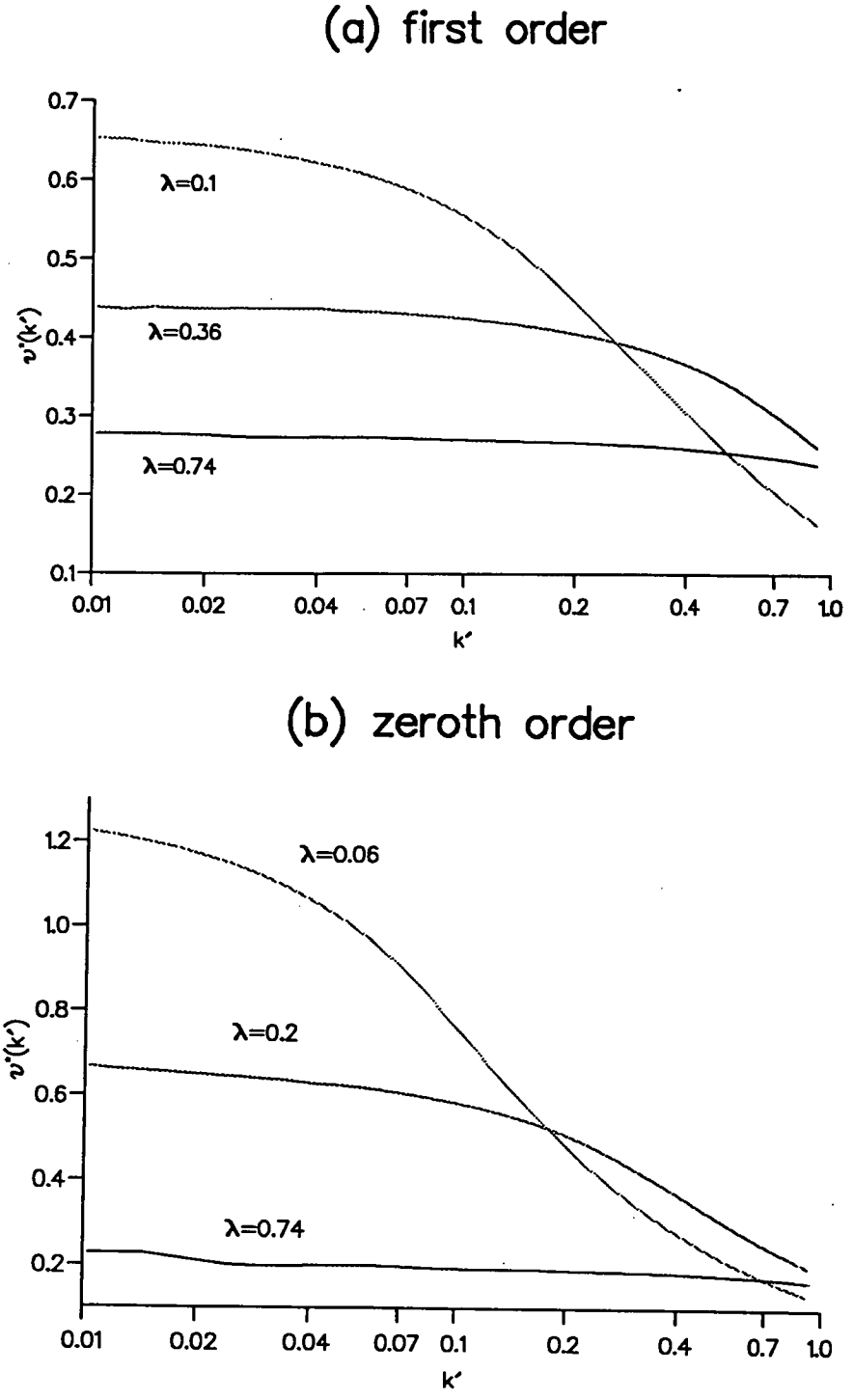

Figure 4.3: Scaled effective viscosity versus scaled wavenumber

It can be noted that the range of the effective viscosity is from 0 to the final upper wavenumber k_N, where the final cycle of the iteration is N. Continuation of the spectrum beyond this cut to check for a power law spectrum for the effective viscosity is not logical since the equations of motion are not defined there.

Also, it can be noted that as $\lambda \rightarrow 0$, the λ dependent behaviour observed in the α v. λ plots sets in at about the same value that the point of inflexion appears in the $\nu^*(k')v.k'$ plot (e.g. the $\lambda = 0.10$ plot for the first order). The significance of this may become more apparent when a mathematical form of the conditional average is derived. I would expect that this change in the behaviour of the effective viscosity is related to the dependence of $u(k_0)$ on the explicit scales once the band narrows to certain size. This may be indicative of a finite correlation length for modes in the inertial range under the conditional average.

The effective viscosity is dependent on λ, over all λ values, and thus it is not possible to argue for the existence of a special value of the parameter from this data. However, the calculation of the Kolmogorov constant clearly indicates a special region, which coincides qualitatively with what one would expect by considering the method of derivation and its assumptions (see also figure 4.7 and the discussion associated).

$\nu_n(k)$ v. k for various cycle numbers

In figure 4.4, the development of the real (unscaled) effective viscosity is charted under the iteration from an initial value of 1. The results presented are for the zeroth order at $\lambda = 0.20$ and the first order at $\lambda = 0.36$, thus lying in the plateau region for both calculations. The final n value given is where the scaled effective viscosity reaches the fixed point.

As more modes are accounted for by the subgrid viscosity so the effective viscosity rises. Also, it is clear how from a starting range of 0 to 1 for k space, at convergence there only remains about a hundredth of this range to be simulated

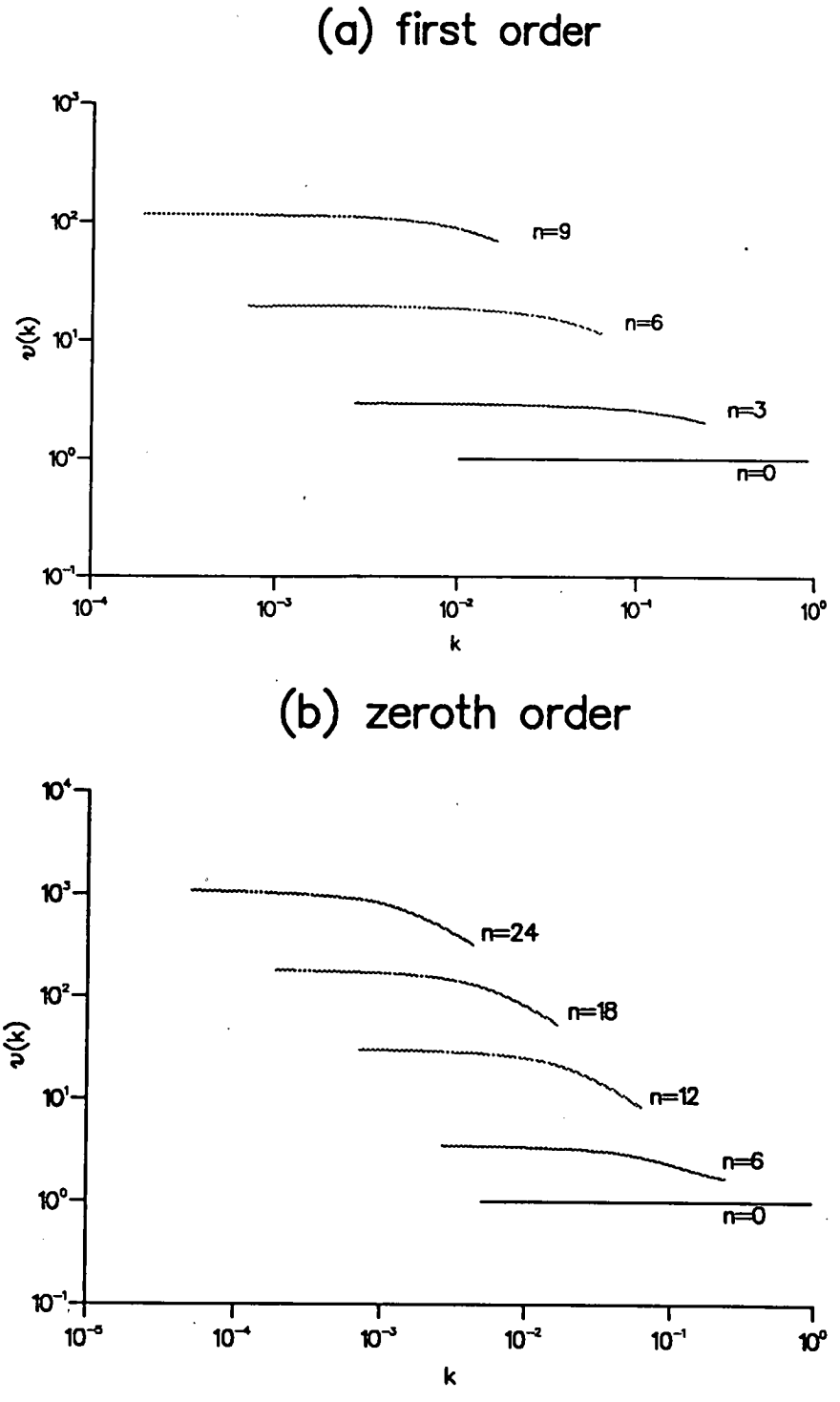

Figure 4.4: Development of effective viscosity under the iteration

thus illustrating the power of the method in reducing the computational task. For simplicity I have assumed $k_0 = 1$.

The roll-off as the cut is approached appears early on in the calculation. The properties of $\nu(k)$ are thus very similar to those of $\nu^*(k')$; the asymptote as $k \to 0$ and the roll-off as $k \to k_N$.

$\tilde{\nu}_n(0)$ and $\tilde{\nu}_n(1)$ v. n

Figure 4.5 shows the development, as the iteration proceeds, of the effective viscosity at either end of the k' range. These graphs are the results of calculations with the first order approximation. As usual, three values of λ are chosen to reflect the three regions of different behaviour. These graphs show an even progression to the fixed point, indicating that the iteration is stable. Although the effect of averaging out large sections of k-space at low λ values makes the discrete nature of the iteration clear.

$\nu^*(1)$ v. n for various ν_0 values

Figure 4.6 shows that the fixed point is valid for a wide range of starting values for the molecular viscosity ($\tilde{\nu}_0$=0.1, 1 and 2). The calculations are from the first order approximation and at a plateau value of the bandwidth, $\lambda = 0.36$. This illustrates the principle of universality where the answers are independent of details of the system. The results depend only on the general dynamics of the inertial range. This fixed point is valid for any Reynolds number provided it is high enough to have an inertial range. This is to be expected since the theory implicitly assumes that viscous effects play no part in the description of the subgrid scales and these results confirm that view.

The alpha values calculated in these runs were, of course, also the same (within tolerance) for the various starting $\tilde{\nu}_0$ values.

$\nu^*(k')$ v. λ for various k' values

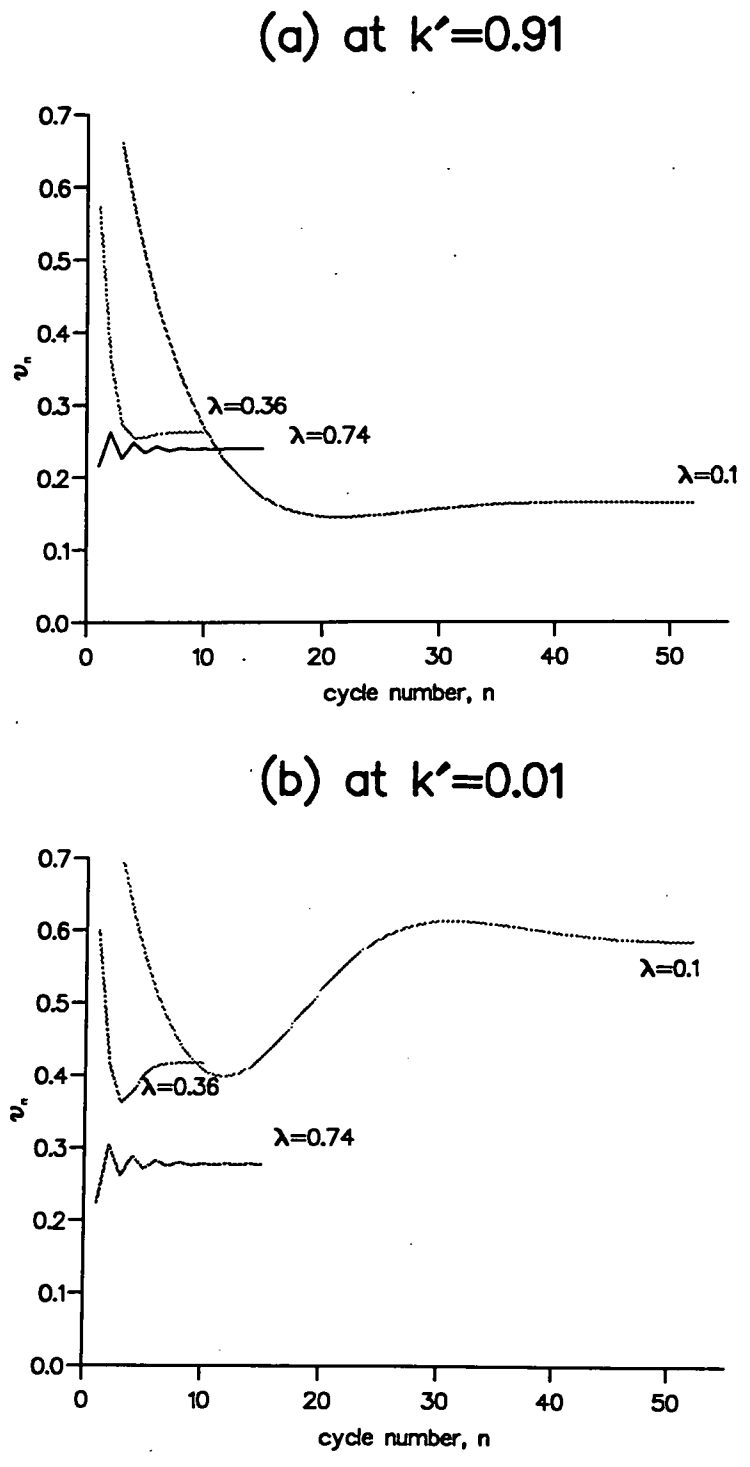

Figure 4.5: Development of the scaled effective viscosity under the iteration

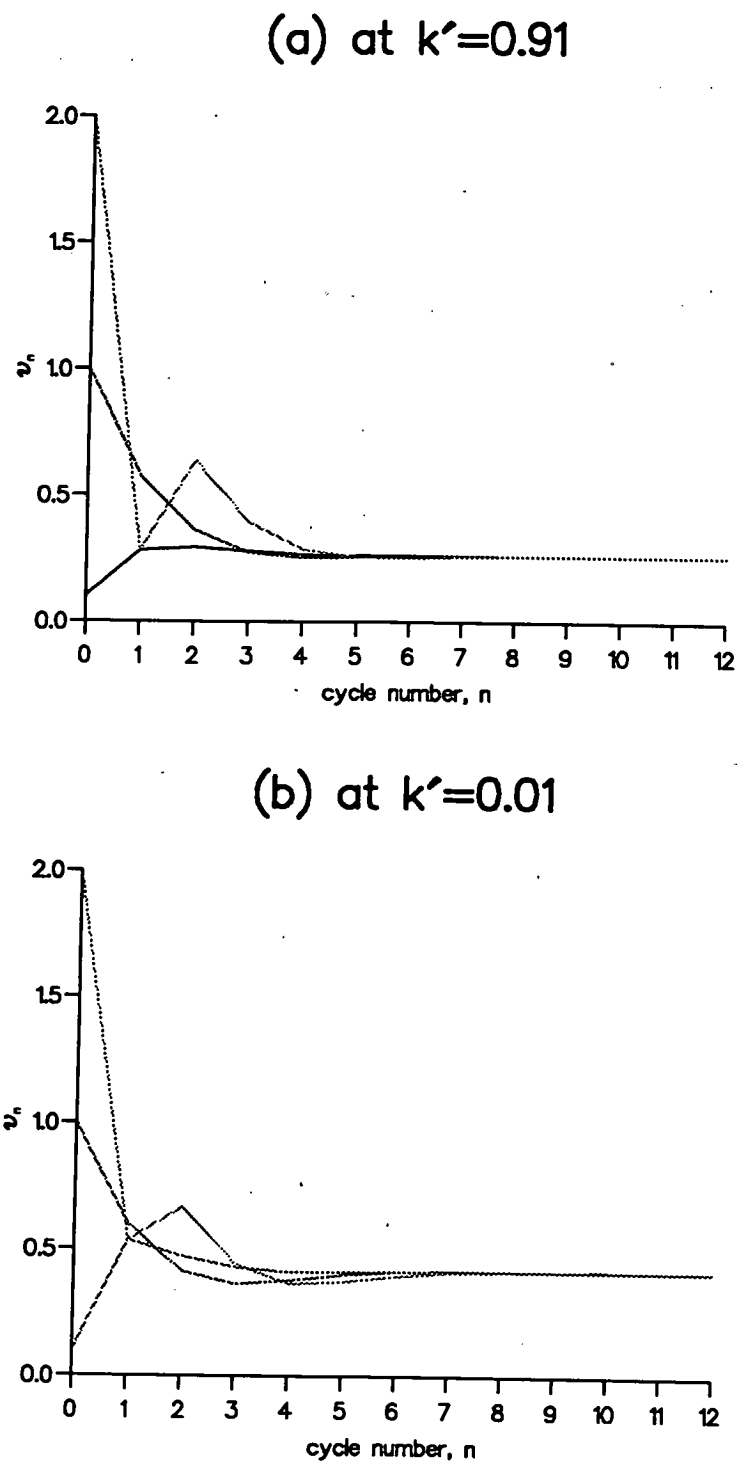

Figure 4.6: Convergence of the scaled viscosity to the fixed point for several initial values of viscosity

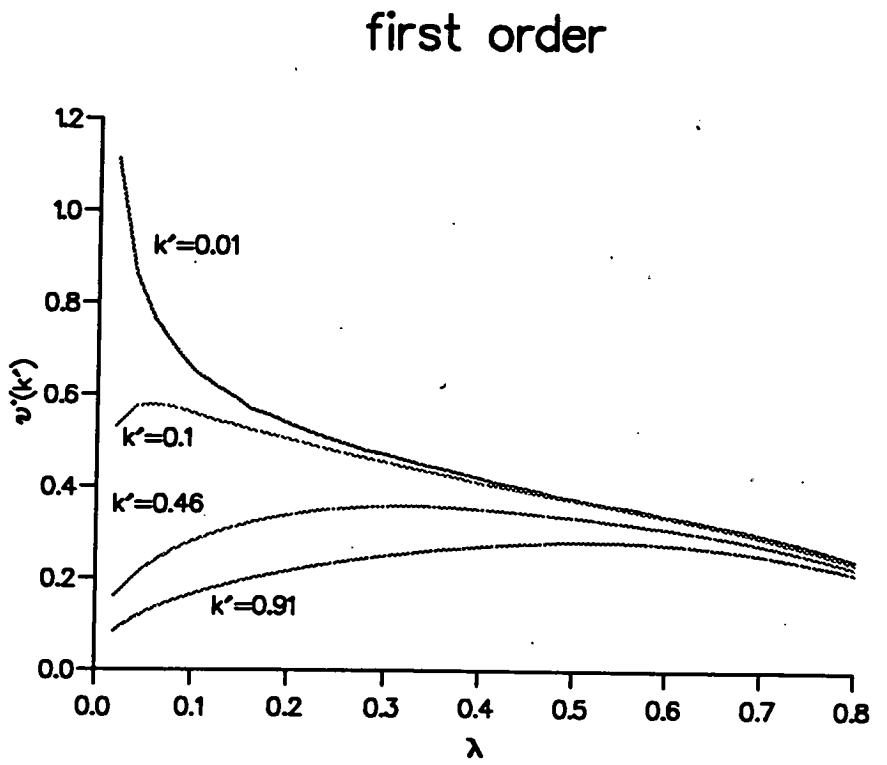

Figure 4.7: Scaled effective viscosity versus bandwidth parameter

Figure 4.7 shows the variation with bandwidth of the effective viscosity, for a few k' values in the first order approximation. This graph shows that the fixed point of the effective viscosity does vary with λ. So the fixed point for the effective viscosity is λ dependent although as the graph shows for the large wavenumbers there is a section, corresponding to the plateau, where it is not a very strong dependence. A criterion for picking a suitable λ value is thus only obtained from the α v. λ plot. This variation at the low wavenumbers ought not to effect any results since the term appearing in the new NSE is $\nu_N(k)k^2$ which will thus weight towards the large k values heavily, where there appears to be less variation. More investigation needs to be done to discern if there are compensating effects at work at different wavenumbers which lead to the plateau in the Kolmogorov constant. Since the theory is approximate there will not be the complete independence from λ expected of an exact theory and thus the best criterion for choosing a parameter value to work with is obtained when the Kolmogorov constant is independent of λ.

In the integral used to calculate α, $\nu^*(k')$ is multiplied by $k'^{1/3}$ and it is clear from the results that this weighting in the integrand compensates for the rise in effective viscosity as $\lambda \rightarrow 0$. (Recall that α is equal to this integral to the power $-2/3$, hence, a falling value of the integral implies a rise in the Kolmogorov constant.) Of course, similarly, as $\lambda \rightarrow 1$ the fall in effective viscosity explains the rise in α values.

As $\lambda \rightarrow 1$, the scaled effective viscosity is tending to a constant and the value of this constant is falling. There is no convergence for λ values greater than those given, even after many hours of running time. This limit represents the case where the band covers all of k-space, thus the effective viscosity is defined only over a small piece of k-space. The rest of the modes are subgrid and thus subject to all the assumptions made in the derivation.

As $\lambda \rightarrow 0$, the scaled effective viscosity is falling over most of its range. This limit represents the case where the band covers none of k-space so there are no subgrid modes and hence no increment to viscosity. Thus the effective viscosity would be expected to remain at its initial value. However, the theory is dependent on a non-zero increment in the formulation and the pathological behaviour in this limit is probably due to numerical inaccuracies as the band shrinks.

k_N v. λ

Figure 4.8 plots the cut-off wavenumber at the final cycle, N, (at which convergence is reached) against the bandwidth parameter. Hence, this measures how much of k-space is averaged away before convergence is obtained; a larger value of k_N means more rapid convergence.

The large variations at low λ values are due to the discrete nature of the iteration procedure. In fact, there is usually only one cycle difference in the final cycle number for these peaks and troughs and an average would look much flatter. For those λ values in the 0.2-0.5 region only about 10 iterations are being done.

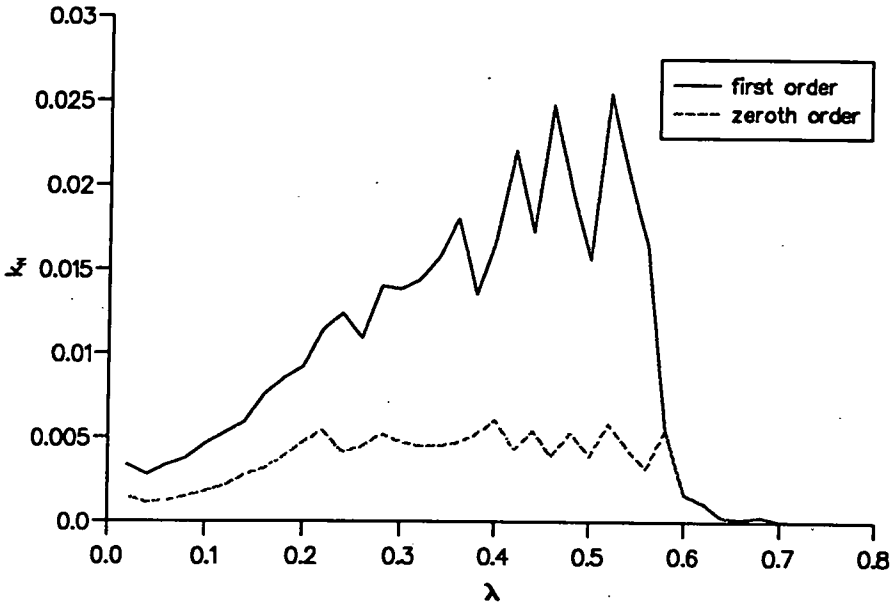

Figure 4.8: Final cut-off wavenumber versus bandwidth parameter

4.5 Application to Large Eddy Simulations

The work described above will supply a form of the effective viscosity to be used in an LES of the modified NSE. The principal condition for the use of the theory here presented is that there is a sufficient Reynolds number to have an inertial range. That inertial range ought not to be too short otherwise the energy-containing range dynamics will be effected by viscous forces and the turbulence will not produce the universal behaviour expected. However, before this can be done the cut-off wavenumber for the simulation will need to be judged. This will be determined by where the starting wavenumber for the inertial range lies, since this effective viscosity is only valid as a description of the behaviour of the modes in the inertial range. However, the fixed point will capture all the behaviour of those high-wavenumber modes and therefore, the LES need only be done with the energy-containing modes resolved as explicit scales.

There is the difficulty of matching the universal (inertial) to the non-universal (energy-containing) ranges. This cannot be done generally, However, for practical purposes provided the non-universal range can be simulated in an LES then the calculation can be extended to any Reynolds' number by decreasing ν_0 since it will be the value of $\nu_n(k)$ from a given cut-off wavenumber k_n that will fix the Reynolds' number of the simulation. In other words, we can fix the upper wavenumber to be what is required by doing enough iterations from k_0.

This theory can also be applied to flow with different geometries provided the assumptions that the inertial range exists and is large enough to allow for the universal behaviour to dominate the dynamics(i.e. the Reynolds number has to be sufficiently large). Implicit is also the assumption that the subgrid modes will be homogeneous and isotropic. This ought to be true for many types of real flow where any appreciable shear is on a long length scale and thus, does not effect the subgrid modes or, as in pipe flow, where the region of high shear is near a boundary where the local Reynolds' number is low anyway.

Chapter 5

Passive Scalar Transport

5.1 Introduction

I shall now consider the application of the method developed in chapter 3 to the problem of the convection of a scalar field by an isotropic, homogeneous turbulent velocity field in an incompressible fluid. This scalar field is assumed to be passive in the sense that its fluctuations do not appreciably affect the physical characteristics of the fluid (e.g. density).

A solution to this problem has important applications as it describes not only the behaviour of a passive contaminant but also the dynamics of a temperature field. For practical purposes, this is useful in the calculation of the propagation of electromagnetic waves in sea water where the salinity fluctuations scatter them or in cases of pollution of a flow.

The first attempt to apply an iterative renormalisation group method to a turbulence problem was by Rose[28] who worked on this problem of passive scalar transport. At first sight, it presents a less arduous challenge than the velocity field problem since the dynamical equation of motion for the passive scalar field,

equation (5.1), is bilinear in the velocity field rather than non-linear. However, it has proved consistently difficult for RPTs which provide good agreement with experiment in the velocity field problem to furnish a similar success with the passive scalar problem. Nevertheless, the following theory obtains feasible results for the physical measurables, such as the Obukhov-Corrsin constant and the turbulent Prandtl number.

The notation used in this chapter will be analogous to that of chapter 3. As in chapter 3, an effective diffusivity will be derived using a conditional average technique, which can be applied to an LES for passive scalar transport. The range of Prandtl and Reynolds numbers for which this result is applicable will be discussed and then a brief description given of how the Obukhov-Corrsin constant is calculated. The code used to perform these calculations will be outlined followed by a presentation and discussion of the results obtained.

5.2 Derivation of the Effective Diffusivity

5.2.1 Rewriting the equation of motion

The dynamical equation for a passive scalar field $\phi(\mathbf{x},t)$, is

$$\frac{\partial}{\partial t}\phi(\mathbf{x},t)+\mathbf{u}(\mathbf{x},t).\nabla\phi(\mathbf{x},t)=\chi_0\nabla^2\phi(\mathbf{x},t) \tag{5.1}$$

where χ_0 is the molecular diffusivity of the scalar field. The velocity field $\mathbf{u}(\mathbf{x},t)$ satisfies the NSE and is statistically homogeneous, isotropic and stationary as before.

As in the velocity field problem, two mechanisms are evident in this equation. On the one hand, there is the forced convection of the scalar by the velocity field and on the other, there is the molecular diffusion of the scalar. The equivalent for the

Reynolds number in the passive scalar transport problem is the Peclet number, which represents the ratio of forced convection effects to diffusion effects. It is given by

$$Pe = \frac{\overline{u}l}{\chi} \tag{5.2}$$

where $\overline{u}$ is an r.m.s. velocity scale and l is a length scale for the scalar field fluctuations.

The cut-off wavenumber above which diffusion dominates the dynamics of the ϕ field is given by the dissipation relation

$$\epsilon_\phi = \int_0^\infty 2\chi_0(k)k^2F(k)dk \doteq \int_0^{k_{\phi 0}} 2\chi_0(k)k^2F(k)dk. \tag{5.3}$$

Corrsin[46] has applied the same type of arguments used in the Kolmogorov analysis of the velocity field and derived a power law for the scalar variance spectrum,

$$F(k) = \beta\epsilon_\phi\epsilon^{-1/3}k^{-5/3}, \tag{5.4}$$

where β is the Obukhov-Corrsin constant. This analysis is only valid in the region of wavenumbers where there is power law for both the velocity and scalar fields. Now further analysis gives that the cut-off wavenumber for the scalar field for the case $\nu \ll \chi$ is on the order

$$k_{\phi 0} = (\epsilon/\chi^3)^{1/4} \tag{5.5}$$

due to Corrsin[46]. For the case $\nu \gg \chi$ on the order

$$k_{\phi 0} = (\epsilon/\nu\chi^2)^{1/4} \tag{5.6}$$

due to Batchelor[47].For the velocity field, we have

$$k_0 = (\epsilon/\nu^3)^{1/4}, \tag{5.7}$$

the Kolmogorov wavenumber. An important ratio is given by the Prandtl number, which is defined as

$$Pr = \frac{\nu}{\chi}. \tag{5.8}$$

For now, it is assumed that the Prandtl number is 1. This implies that the cut-off wavenumbers are equal and hence simplifies the analysis. The Reynolds number and Prandtl number dependence of this derivation will be discussed in the Comments section.

Following the procedure used in the derivation of an effective viscosity an effective diffusivity can be derived. Splitting the velocity and scalar fields into two parts, thus defining the super- and sub- grid scales:

$$\phi(\mathbf{k}) = \begin{cases} \phi^-(\mathbf{k}) & \text{for } 0 \le k \le k_1 \\ \phi^+(\mathbf{k}) & \text{for } k_1 \le k \le k_0, \end{cases} \tag{5.9}$$

where k_0 is defined in (3.1) and $k_1 = (1-\lambda)k_0$ and

$$u_\alpha(\mathbf{k}) = \begin{cases} u_\alpha^-(\mathbf{k}) & \text{for } 0 \le k \le k_1 \\ u_\alpha^+(\mathbf{k}) & \text{for } k_1 \le k \le k_0. \end{cases} \tag{5.10}$$

It can be noted that the scalar field is divided in the same way as the velocity field.

A conditional average is postulated over the subensemble of all ϕ and $\mathbf{u}$ fields satisfying (5.1) and the NSE respectively, such that ϕ^- and $\mathbf{u}^-$ are constant in all elements of the subensemble. This represents an extension of the average introduced in chapter 3 by including the ensemble of scalar fields into the turbulent ensemble. The new average places a restriction on the partial ensemble by selecting only those members of this dual ensemble which have both scalar and velocity explicit scale fields constant from member to member of the ensemble. This average can be written as

$$A[\phi^+\mathbf{u}^+|\phi^-\mathbf{u}^-]u_\alpha \dots u_\gamma\phi \dots \phi = \langle u_\alpha \dots u_\gamma\phi \dots \phi\rangle_0. \tag{5.11}$$

Hence this operation has the properties

$$\langle u_\alpha^-(\mathbf{k},t)\rangle_0 = u_\alpha^-(\mathbf{k},t) \tag{5.12}$$

and

$$\langle \phi^-(\mathbf{k},t)\rangle_0 = \phi^-(\mathbf{k},t). \tag{5.13}$$

As for the velocity field, a further decomposition of these two fields is introduced, which identifies the component of the subgrid scales unaffected by this conditional average. The new fields are

$$u_\alpha^+(\mathbf{k},t) = v_\alpha^+(\mathbf{k},t) + \Delta_\alpha^+(\mathbf{k},t) \tag{5.14}$$

and

$$\phi^+(\mathbf{k},t) = \psi^+(\mathbf{k},t) + \gamma^+(\mathbf{k},t). \tag{5.15}$$

The crucial properties of the $\mathbf{v}^+$ and ψ^+ fields are that they are independent of $\mathbf{u}^-$ and ϕ^- and consequently, any conditional average acts as a global average on these fields. Also, they have the same properties of homogeneity, isotropy and stationarity under the normal turbulent average as $\mathbf{u}^+$ and ϕ^+, respectively. The following equations will prove useful in the later analysis; by homogeneity of the average of the $\mathbf{v}^+$ field,

$$\langle v_\alpha^+(\mathbf{k},t)\rangle_0 = \langle v_\alpha^+(\mathbf{k},t)\rangle = 0; \tag{5.16}$$

by homogeneity of the average of the ψ^+ field,

$$\langle \psi^+(\mathbf{k},t)\rangle_0 = \langle \psi^+(\mathbf{k},t)\rangle = 0; \tag{5.17}$$

by homogeneity, isotropy and stationarity of the average of the $\mathbf{v}^+$ field,

$$\langle v_\alpha^+(\mathbf{k},t) v_\beta^+(\mathbf{k}',t)\rangle = Q_v^+(k) D_{\alpha\beta}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}'); \tag{5.18}$$

and by the isotropy of the average of the $\mathbf{v}^+$ field,

$$\langle v_\alpha^+(\mathbf{k},t)\psi^+(\mathbf{j},t)\rangle_0 = \langle v_\alpha^+(\mathbf{k},t)\psi^+(\mathbf{j},t)\rangle = 0. \tag{5.19}$$

In order to ease mathematical manipulations, the work will be carried out in Fourier space, so transforming the fields according to (1.5) and

$$\phi(\mathbf{x},t) = \int d^3k \phi(\mathbf{k},t) exp(i\mathbf{k}.\mathbf{x}) \tag{5.20}$$

we obtain

$$\left[\frac{\partial}{\partial t} + \chi_0 k^2\right] \phi(\mathbf{k},t) = -ik_\alpha \int d^3j u_\alpha(\mathbf{k}-\mathbf{j})\phi(\mathbf{j},t). \tag{5.21}$$

Now using these tools, the equation of motion for the explicit scale behaviour of the scalar field will be rewritten. This new equation of motion is obtained by applying the conditional average to (5.21),

$$L_0\phi^-(\mathbf{k},t) = -ik_\alpha \int d^3j \langle u_\alpha(\mathbf{k}-\mathbf{j},t)\phi(\mathbf{j},t)\rangle_0 - \langle L_0\phi^+(\mathbf{k},t)\rangle_0, \tag{5.22}$$

where

$$L_0 = \left[\frac{\partial}{\partial t} + \chi_0 k^2\right]. \tag{5.23}$$

The equation for the subgrid modes is obtained by subtracting (5.22) from (5.21),

$$L_0\phi^+(\mathbf{k},t) = -ik_\alpha \int d^3j \left[u_\alpha(\mathbf{k}-\mathbf{j},t)\phi(\mathbf{j},t) - \langle u_\alpha(\mathbf{k}-\mathbf{j},t)\phi(\mathbf{j},t)\rangle_0\right] + \langle L_0\phi^+(\mathbf{k},t)\rangle_0. \tag{5.24}$$

Now, as before, the equation for the supergrid modes will be recast with an incremented diffusivity so that it has the same form as (5.21). The main task is then to find a form for the new diffusivity and iterate the diffusivity to see if a fixed point exists. Thus, the problem breaks down into parts; finding an expression for the increment to diffusivity in the first shell; scaling the new dynamical equation and repeating this procedure until convergence for the diffusivity, and hence the dynamical equation, is reached.

Using the conditional properties of the average, the explicit scale equation becomes

$$\begin{aligned} L_0\phi^-(\mathbf{k},t) + \langle ik_\alpha \int d^3j u_\alpha^+(\mathbf{k}-\mathbf{j},t)\phi^+(\mathbf{j},t)\rangle_0 &= -ik_\alpha \int d^3j u_\alpha^-(\mathbf{k}-\mathbf{j},t)\phi^-(\mathbf{j},t) \\ &\quad - J(\mathbf{k},t), \end{aligned} \tag{5.25}$$

where

$$\begin{aligned} J(\mathbf{k},t) &= ik_\alpha \int d^3j \langle u_\alpha^-(\mathbf{k}-\mathbf{j},t)\phi^+(\mathbf{j},t)\rangle_0 + ik_\alpha \int d^3j \langle u_\alpha^+(\mathbf{k}-\mathbf{j},t)\phi^-(\mathbf{j},t)\rangle_0 \\ &\quad + \langle L_0\phi^+(\mathbf{k},t)\rangle_0. \end{aligned} \tag{5.26}$$

The same procedure applied to equation (5.23) results in the equation for the implicit scales,

$$\begin{aligned} L_0\phi^+(\mathbf{k},t) &= -ik_\alpha \int d^3j\{u_\alpha^+(\mathbf{k}-\mathbf{j},t)\phi^-(\mathbf{j},t) + u_\alpha^-(\mathbf{k}-\mathbf{j},t)\phi^+(\mathbf{j},t) \\ &\quad + u_\alpha^+(\mathbf{k}-\mathbf{j},t)\phi^+(\mathbf{j},t) - \langle u_\alpha^+(\mathbf{k}-\mathbf{j},t)\phi^+(\mathbf{j},t)\rangle_0\} \\ &\quad + J(\mathbf{k},t). \end{aligned} \tag{5.27}$$

The term $J(\mathbf{k},t)$ may be re-expressed using the new fields as,

$$\begin{aligned} J(\mathbf{k},t) &= ik_\alpha \int d^3j u_\alpha^-(\mathbf{k}-\mathbf{j},t)\langle\gamma^+(\mathbf{j},t)\rangle_0 + ik_\alpha \int d^3j \langle\Delta_\alpha^+(\mathbf{k}-\mathbf{j},t)\rangle_0\phi^-(\mathbf{j},t) \\ &\quad + L_0\langle\gamma^+(\mathbf{k},t)\rangle_0. \end{aligned} \tag{5.28}$$

This will be small provided that the $\mathbf{v}^+(\mathbf{k},t)$ and $\psi^+(\mathbf{k},t)$ fields are a good approximation to the real subgrid fields under the conditional average. The terms in $J(\mathbf{k},t)$ depend on the degree of coupling between modes inside and those outside the band $k_1 \leq k \leq k_0$. As these modes are in the range close to the dissipation/diffusive region then it is to be expected that coupling will be local and small. The precise error involved in neglecting $J(\mathbf{k},t)$ will not be clear until some choice is made for the $\mathbf{v}^+(\mathbf{k},t)$ and $\psi^+(\mathbf{k},t)$ fields.

I will now move to consider the second term on the LHS of (5.25), which will provide the increment to the diffusivity. If the decomposition of (5.14) and (5.15) is applied then the mixed moment becomes

$$\begin{aligned} \langle u_\alpha^+(\mathbf{k}-\mathbf{j},t)\phi^+(\mathbf{j},t)\rangle_0 &= \langle v_\alpha^+(\mathbf{k}-\mathbf{j},t)\psi^+(\mathbf{j},t)\rangle_0 + \langle\Delta_\alpha^+(\mathbf{k}-\mathbf{j},t)\psi^+(\mathbf{j},t)\rangle_0 \\ &\quad + \langle v_\alpha^+(\mathbf{k}-\mathbf{j},t)\gamma^+(\mathbf{j},t)\rangle_0 + \langle\Delta_\alpha^+(\mathbf{k}-\mathbf{j},t)\gamma^+(\mathbf{j},t)\rangle_0. \end{aligned} \tag{5.29}$$

Now, the leading term in the decoupled fields vanishes due to isotropy. Therefore, consideration will shift to the expression for the whole term in (5.29) to see if a re-expansion of it can yield a term leading in the decoupled fields. The equation

for this term is

$$\begin{aligned}\langle\phi^+(\mathbf{j},t)u_\alpha^+(\mathbf{l},t)\rangle_0 \;=\; & \int_{-\infty}^{t} dt' exp(-[\chi_0 j^2+\nu_0 l^2])(t-t')\times \\ & \{-ij_\beta\int d^3\mathbf{p}[\langle\phi^-(\mathbf{p},t')u_\beta^+(\mathbf{j}-\mathbf{p},t')u_\alpha^+(\mathbf{l},t')\rangle_0 \\ & \quad+\langle\phi^+(\mathbf{p},t')u_\beta^-(\mathbf{j}-\mathbf{p},t')u_\alpha^+(\mathbf{l},t')\rangle_0 \\ & \quad+\langle\phi^+(\mathbf{p},t')u_\beta^+(\mathbf{j}-\mathbf{p},t')u_\alpha^+(\mathbf{l},t')\rangle_0 \\ & \quad-\langle\langle\phi^+(\mathbf{p},t')u_\beta^+(\mathbf{j}-\mathbf{p},t')\rangle_0 u_\alpha^+(\mathbf{l},t')\rangle_0] \\ & +\langle u_\alpha^+(\mathbf{l},t')J(\mathbf{j},t)\rangle_0 \\ & +M_{\alpha\beta\gamma}(\mathbf{l})\int d^3\mathbf{p}[2u_\beta^-(\mathbf{p},t')\langle\phi^+(\mathbf{j},t')u_\gamma^+(\mathbf{l}-\mathbf{p},t')\rangle_0 \\ & \quad+\langle u_\beta^+(\mathbf{p},t')\phi^+(\mathbf{j},t')u_\gamma^+(\mathbf{l}-\mathbf{p},t')\rangle_0 \\ & \quad-\langle\langle u_\beta^+(\mathbf{p},t')u_\gamma^+(\mathbf{l}-\mathbf{p},t')\rangle_0\phi^+(\mathbf{j},t')\rangle_0] \\ & -\langle\phi^+(\mathbf{j},t')H_\alpha(\mathbf{l},t)\rangle_0\}, \end{aligned} \tag{5.30}$$

where $\mathbf{l}=\mathbf{k}-\mathbf{j}$. Once again, we are faced with the problem of solving for ever more complex moments when the distribution of the velocity and scalar fields under the conditional average is unknown. However, the introduction of the new fields provides a truncation as follows,

$$\begin{aligned}\langle\phi^+(\mathbf{j},t)u_\alpha^+(\mathbf{l},t)\rangle_0 \;=\; & \int_{-\infty}^{t} dt' exp(-[\chi_0 j^2+\nu_0 l^2])(t-t')\times \\ & \{-ij_\beta\int d^3\mathbf{p}[\phi^-(\mathbf{p},t')\langle v_\beta^+(\mathbf{j}-\mathbf{p},t')v_\alpha^+(\mathbf{l},t')\rangle_0 \\ & \quad+\phi^-\langle\mathbf{v}^+\boldsymbol{\Delta}^+\rangle_0+\phi^-\langle\boldsymbol{\Delta}^+\mathbf{v}^+\rangle_0+\phi^-\langle\boldsymbol{\Delta}^+\boldsymbol{\Delta}^+\rangle_0 \\ & \quad+\langle\psi^+\mathbf{u}^-\mathbf{v}^+\rangle_0+\langle\psi^+\mathbf{u}^-\boldsymbol{\Delta}^+\rangle_0 \\ & \quad+\langle\gamma^+\mathbf{u}^-\mathbf{v}^+\rangle_0+\langle\gamma^+\mathbf{u}^-\boldsymbol{\Delta}^+\rangle_0 \\ & \quad+\langle(\psi^++\gamma^+)(\mathbf{v}^++\boldsymbol{\Delta}^+)(\mathbf{v}^++\boldsymbol{\Delta}^+)\rangle_0 \\ & \quad-\langle\phi^+(\mathbf{p},t')u_\beta^+(\mathbf{j}-\mathbf{p},t')\rangle_0\langle\Delta_\alpha^+(\mathbf{l},t')\rangle_0] \\ & +\langle\Delta_\alpha^+(\mathbf{l},t')\rangle_0 J(\mathbf{j},t) \\ & +M_{\alpha\beta\gamma}(\mathbf{l})\int d^3\mathbf{p}[2u_\beta^-(\mathbf{p},t') \\ & \quad(\langle\psi^+\mathbf{v}^+\rangle_0+\langle\psi^+\boldsymbol{\Delta}^+\rangle_0+\langle\gamma^+\mathbf{v}^+\rangle_0+\langle\psi^+\boldsymbol{\Delta}^+\rangle_0) \\ & \quad+\langle(\mathbf{v}^++\boldsymbol{\Delta}^+)(\psi^++\gamma^+)(\mathbf{v}^++\boldsymbol{\Delta}^+)\rangle_0 \end{aligned}$$

$$+ \langle u_\beta^+(\mathbf{p},t')u_\gamma^+(\mathbf{l}-\mathbf{p},t')\rangle_0\langle\gamma^+(\mathbf{j},t')\rangle_0]$$
$$- \langle\gamma^+(\mathbf{j},t')\rangle_0 H_\alpha(\mathbf{l},t')\}. \qquad (5.31)$$

Now to simplify this terms of $O(\langle\Delta^+\rangle_0)$ and $O(\langle\gamma^+\rangle_0)$ will be dropped as small. This argument is similar to that in chapter 3 where it was only the Δ^+ field that was dealt with whereas here we neglect all $O(\langle\Delta^+\rangle_0)$ and $O(\langle\gamma^+\rangle_0)$ terms. This will be further discussed in the Comments section.

The hierarchy generating terms (i.e. those of the form $\langle\phi^+\mathbf{v}^+\mathbf{v}^+\rangle_0$) are neglected. These terms will produce higher order correlations between the decoupled fields but with a corresponding increase in the number of band integrations to be done when these are scaled. This will lead to a higher order dependence on the band width. A greater dependence on the band width will lead to these terms being less significant at the fixed point.

At this point, the obvious difference between the passive scalar transport problem and that of the velocity field is the presence of terms with factors of $\mathbf{u}^-$ in equation (5.31). These vanish in the leading term of the decoupled fields due to isotropy and will be neglected. This is unlike the velocity field case where certain terms vanished due to the form of the increment term (i.e. the presence of an $M(\mathbf{k})$ type factor). Here the vanishing is due to the $\langle\psi^+\mathbf{v}^+\rangle_0$ factor which is zero not a delta function.

However, the theory might be accused of being inconsistent at this point since a term of this form is exactly that which appears initially in equation (5.25) as the increment term and is not dropped at this point. Of course, to abandon the calculation at this point gives no increment and hence the leading contribution is calculated from the dynamical equation for that mixed moment, (5.31).

If we consider the leading order contribution from the decoupled fields to this $\mathbf{u}^-\phi^+\mathbf{u}^+$ term. The leading order term is dependent on $\mathbf{u}^-\phi^-$. However, there

are now several constraints on the wavenumber arguments present in this term which lead to the volume of integration for the term being roughly a factor of 10 down on that in the term which is retained. There are also two band integrations to be done, which give the term a higher order dependence on the bandwidth parameter. After scaling, this term will be smaller than the term retained which has the factor $\phi^-(\mathbf{p},t')\langle v_\beta^+(\mathbf{j}-\mathbf{p},t')v_\alpha^+(\mathbf{l},t')\rangle$.

5.2.2 The Markovian Approximation

The time integration involved in solving for the higher moments is treated by a Markovian approximation where it is assumed that the explicit modes operate on a time scale that is long compared to that present in the exponential argument $(\chi_0 j^2+\nu_0 l^2)^{-1}$. Thus it is appropriate to replace all explicit terms' time arguments with t, the same as that of the mixed moment being solved.

Considering this in more detail; rewrite the term which is the increment to diffusivity, considering only the leading order identified above,

$$\begin{aligned}\langle ik_\alpha \int d^3j u_\alpha^+(\mathbf{k}-\mathbf{j},t)\phi^+(\mathbf{j},t)\rangle_0 &= ik_\alpha \int d^3j \int_{-\infty}^{t} dt' exp(-\omega_{\phi 2}(j,l)(t-t'))\times \\ &\{-ij_\beta \int d^3\mathbf{p}[\phi^-(\mathbf{p},t')\langle v_\beta^+(\mathbf{j}-\mathbf{p},t')v_\alpha^+(\mathbf{l},t')\rangle_0\},\end{aligned} \tag{5.32}$$

where $\omega_{\phi 2}(j,l)=(\chi_0 j^2+\nu_0 l^2)$. Using the fact that for the $\mathbf{v}^+$ field the conditional average has the same effect as the normal average and that this has the same properties as the global average of $\mathbf{u}^+$ (i.e. homogeneous, isotropic and stationary), then

$$\begin{aligned}&\langle ik_\alpha \int d^3j u_\alpha^+(\mathbf{k}-\mathbf{j},t)\phi^+(\mathbf{j},t)\rangle_0 = \\ &\quad ik_\alpha \int d^3j \int_{-\infty}^{t} dt' exp(-\omega_{\phi 2}(j,l))(t-t')\{-ij_\beta \phi^-(\mathbf{k},t')D_{\beta\alpha}(\mathbf{l})Q_v^+(l)\}\end{aligned} \tag{5.33}$$

In order to solve this integral, the time dependence of the explicit scales must be known. However, in order that the derivation of the increment to the diffusivity only depends on the subgrid scales, this integral will be treated in the context of the separation of of time-scales for the explicit and subgrid scales. If we Taylor expand $\phi^-(\mathbf{k}, t-\tau)$ around $\tau = 0$ we get

$$\langle ik_\alpha \int d^3j u^+_\alpha(\mathbf{k}-\mathbf{j},t)\phi^+(\mathbf{j},t)\rangle_0 =$$
$$ik_\alpha \int d^3j \{-ij_\beta \frac{\phi^-(\mathbf{k},t)}{\omega_{\phi 2}(j,l)} D_{\beta\alpha}(\mathbf{l}) Q^+_v(l) + \text{ higher order time derivatives of } \phi^-\}. \quad (5.34)$$

The higher order terms of (5.34) arise from

$$\int_0^\infty exp(-\omega_{\phi 2}(j,l)\tau)\phi^-(\mathbf{k},t-\tau)d\tau$$
$$= \int_0^\infty exp(-\omega_{\phi 2}(j,l)\tau)\left[\phi^-(\mathbf{k},t) - \tau\frac{\partial\phi^-(\mathbf{k},t)}{\partial t} + O(\tau^2)\right] d\tau \quad (5.35)$$

and thus will involve the decay rate of explicit scale modes and $\omega_{\phi 2}(j,l)$ (decay of two subgrid modes). Hence, the Markovian approximation of the neglect of time histories is equivalent to taking the zeroth order term in a Taylor series for ϕ^- about t.

The time scale for diffusion of the mode $\phi(\mathbf{k},t)$ in a low Peclet number regime is $(\chi_0 k^2)^{-1}$. Thus, in the near diffusion region of the band of wavenumbers averaged out this is reasonable measure of the time scale for decay. However, the explicit scales are driven by much longer time scales those of the forcing terms of the velocity field. Therefore, it is to be expected that the explicit scales are of a different and longer time scale than the subgrid scales. This effect should be most pronounced at the fixed point when there will be genuine separation of the physics of the non-universal from the universal.

The action of the turbulence and the forcing will be to create more organisation and hence a persistence of structure, which in the higher (more randomised) wavenumber modes will not be as important as diffusion.

Hence, the key equation for the explicit scales can be written as

$$\left[\frac{\partial}{\partial t} + (\chi_0 + \delta\chi_0)k^2\right]\phi^-(\mathbf{k},t) = -ik_\alpha \int d^3j u_\alpha^-(\mathbf{k}-\mathbf{j},t)\phi_\gamma^-(\mathbf{j},t). \tag{5.36}$$

5.2.3 The first shell elimination

Thus, the equation for the new diffusivity is

$$\chi_1 = \chi_0 + \delta\chi_0, \tag{5.37}$$

where the formula for the increment to diffusivity is

$$\delta\chi_0(k) = \frac{1}{k^2}\int d^3j \frac{D_{\beta\alpha}(\mathbf{k}-\mathbf{j})k_\alpha j_\beta Q_v^+(|\mathbf{k}-\mathbf{j}|)}{\chi_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2}. \tag{5.38}$$

with

$$0 \le k \le k_1; k_1 \le j, |\mathbf{k}-\mathbf{j}| \le k_0,$$

where Q_v^+ is merely an extension of the spectral density defined by (3.9).

It must be noted at this point that the increment to diffusivity involves the viscosity. Consequently, the calculation of the effective diffusivity must be done in parallel with that for the effective viscosity (see section 5.5).

Now, consider the principal approximation of the theory - the form of these independent fields v^+ and ψ^+. There is no need to introduce an approximation for the ψ^+ field as it is only the v^+ field that appears at this level of approximation in the expression for the increment to diffusivity. Hence, using a Taylor series based at the upper cut-off, k_0, as in chapter 3;

$$v_\alpha^+(\mathbf{k},t) = u_\alpha^+(\mathbf{k_0},t)\text{- the zeroth order approximation,} \tag{5.39}$$

or

$$v_\alpha^+(\mathbf{k},t) = u_\alpha^+(\mathbf{k_0},t) + (\mathbf{k}-\mathbf{k_0}).\nabla u_\alpha^+(\mathbf{j},t)|_{j=k_0} \text{- the first order approximation.} \tag{5.40}$$

It ought to be noted again that the expansion is based on $(\mathbf{k}-\mathbf{k_0})$, which has an upper bound of λk_0.

We may evaluate Q_v^+ , using the equations above, with the result that the increment to the diffusivity takes the form

$$\delta\chi_0(k) = \frac{1}{k^2}\int d^3j \frac{D_{\beta\alpha}(\mathbf{k}-\mathbf{j})k_\alpha j_\beta\{Q(l)|_{l=k_0} + (l-k_0)\frac{\partial Q(l)}{\partial l}|_{l=k_0}\}}{\chi_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2}, \tag{5.41}$$

where $l = |\mathbf{k}-\mathbf{j}|$. Hence, we have succeeded in getting the increment into the form we need to be able to perform the renormalisation transformation on the dynamical equation and that completes the elimination of the first shell of modes. We extend the procedure to further shells, as follows:

(a) Set $u_\alpha^-(\mathbf{k},t) = u_\alpha(\mathbf{k},t)$ and $\phi^-(\mathbf{k},t) = \phi(\mathbf{k},t)$ in equation (5.36), so that we now have a new equation with effective diffusivity $\chi_1(k)$ for Fourier modes on the interval $0 < k < k_1$.

(b) Make the decomposition of (5.9) and (5.10), but this time at $k = k_2$, such that the supergrid modes are now defined in the band $k_2 \leq k \leq k_1$.

(c) Repeat the procedures used to eliminate the first shell of modes in order now to eliminate modes in the band $k_2 \leq k \leq k_1$.

In this way, we can progressively eliminate the effect of high wavenumbers in a series of bands $k_{n+1} < k < k_n$, where

$$k_n = (1-\lambda)^n k_0 \; ; \; 0 \leq \lambda \leq 1, \tag{5.42}$$

with, by induction, the recursion relation for the effective viscosity given by

$$\chi_{n+1}(k) = \chi_n(k) + \delta\chi_n(k), \tag{5.43}$$

where the increment of order n takes the form

$$\delta\chi_n(k) = \frac{1}{k^2}\int d^3j \frac{D_{\beta\alpha}(\mathbf{k}-\mathbf{j})k_\alpha j_\beta Q_v^+(|\mathbf{k}-\mathbf{j}|)}{\chi_n(j)j^2 + \nu_n(|\mathbf{k}-\mathbf{j}|)|\mathbf{k}-\mathbf{j}|^2}. \tag{5.44}$$

for wavenumbers

$$0 \leq k \leq k_{n+1}; k_{n+1} \leq j, |\mathbf{k}-\mathbf{j}| \leq k_n.$$

At the same time, the evolution equation for the explicit scales may be written (inductively, from equation (5.36)), as

$$\left[\frac{\partial}{\partial t} + \chi_n(k)k^2\right]\phi^-(\mathbf{k},t) = -ik_\alpha \int d^3j u_\alpha^-(\mathbf{k}-\mathbf{j},t)\phi_\gamma^-(\mathbf{j},t) \tag{5.45}$$

for wavenumbers in the bands defined above.

5.2.4 The Scaled Equations for Effective Diffusivity

In order to carry out the iteration to a fixed point we must scale the new dynamical equation so that it looks like the original equation (5.21). First of all, note that ϕ is assumed to have the same behaviour as $\mathbf{u}$ in so far as the significant wavenumbers are k_{n+1}, and thus k_0. The other physically significant quantities for the effective diffusivity are the dissipation rates for the scalar variance and the energy, ϵ_ϕ and ϵ.

The scaling is found by looking at the dissipation relation in the nth cycle

$$\epsilon_\phi = \int_0^{k_n} 2\chi_n(k)k^2F(k)dk, \tag{5.46}$$

where $F(k)$ is the spectrum of scalar variance,

$$F(k) = 4\pi k^2\langle\phi(\mathbf{k})\phi(-\mathbf{k})\rangle. \tag{5.47}$$

Using dimensional analysis and the assumption that $F(k)$ only depends on ϵ_ϕ, ϵ and k (i.e. the inertial-convective form), the expression for the scalar spectrum is

$$F(k) = \beta\epsilon_\phi\epsilon^{-1/3}k^{-5/3}. \tag{5.48}$$

From equations (5.43) and (5.44), it can be seen that the effective diffusivity will scale like

$$\chi(k_n k') \sim \epsilon^{1/3}k_n^{-4/3}\tilde{\chi}_n(k'). \tag{5.49}$$

For convenience, the constant will be chosen so that

$$\chi(k_n k') = \alpha^{1/2}\epsilon^{1/3}k_n^{-4/3}\tilde{\chi}_n(k'). \tag{5.50}$$

The convenience of this choice is evident when the increment is scaled, thus

$$\begin{aligned}\delta\chi_n(k) &= \frac{1}{k^2}\int d^3j \frac{D_{\beta\alpha}(\mathbf{k}-\mathbf{j})k_\alpha j_\beta Q_v^+(|\mathbf{k}-\mathbf{j}|)}{\chi_n j^2 + \nu_n|\mathbf{k}-\mathbf{j}|^2} \\ &= \alpha^{1/2}\epsilon^{1/3}k_{n+1}^{-4/3}h^{-4/3}\delta\tilde{\chi}_n(k').\end{aligned} \tag{5.51}$$

Hence, the increment and the effective diffusivity scale in the same way. The scaled increment is

$$\delta\tilde{\chi}_n(k') = \frac{1}{4\pi k'^2}\int d^3j' \frac{D_{\beta\alpha}(\mathbf{k}'-\mathbf{j}')k'_\alpha j'_\beta Q'}{\tilde{\chi}_n(hj')j'^2 + \tilde{\nu}_n(hl')l'^2} \tag{5.52}$$

for wavenumbers

$$0 \le k' \le 1; 1 \le j', l' \le h^{-1}$$

where $l' = |\mathbf{k}'-\mathbf{j}'|$ and Q' depends on our approximation for the v^+ field, as in chapter 3.

The scaled recursion relation can now be derived from (5.43), as in Appendix C for the effective viscosity,

$$\tilde{\chi}_{n+1}(k') = h^{4/3}\tilde{\chi}_n(hk') + h^{-4/3}\delta\tilde{\chi}_n(k'). \tag{5.53}$$

This equation is now used in an iteration to find a fixed point for the effective diffusivity, such that

$$\tilde{\chi}_{n+1}(k') = \tilde{\chi}_n(k') = \chi^*(k'). \tag{5.54}$$

The results of these calculations are given in section 5.5, along with the calculation of the Obukhov-Corrsin constant.

5.3 Comments

It is now necessary to discuss the assumptions and approximations that have lead to the above theory. Firstly, a field ψ^+ must exist and have the properties that $\langle\psi^+\rangle_0 \doteq \langle\phi^+\rangle_0$ so that J is small and that ψ^+ is independent of ϕ^- and $\mathbf{u}^-$. This will be the case if a range, equivalent to the inertial range in the velocity field problem, exists where the behaviour of the scalar is dependent only on local dynamics not dissipation. Thus, there will be a power law region for the scalar variance spectrum dependent on ϵ_ϕ, ϵ and the wavenumber k. The interactions in this region will be independent of the non-universal factors since the turbulence will have scrambled the structure of the low wavenumbers. This range is called the inertial-convective range. It is 'convective' because the scalar fields dynamics are dominated by convection by the velocity field.

Secondly, the Reynolds number must be large enough for the existence of an inertial range as must the Peclet number. Thus, the conditions for the existence of an inertial-convective range will be satisfied. The wavenumber characterising diffusion, $k_{\phi 0}$, must lie sufficiently above the wavenumber indicating the onset of universal behaviour for a '-5/3' spectrum of scalar variance to exist. Provided these criteria are satisfied then the value of the Prandtl number is irrelevant, since there will always exist a large '-5/3' region above the non-universal scales whose fixed point will dominate the dynamics. It has been assumed that the Prandtl

number is 1 and consequently, the diffusive cut-off, $k_{\phi 0}$, is equal to the dissipative cut-off, k_0. It will now be argued that this is unnecessary.

If $Pr \gg 1$ then there will exist a range in which convection of the scalar field dominates while viscous forces dominate the velocity field behaviour. However, our iteration begins at the low wavenumber end of the dissipation range and consequently we will only ever be in a '-5/3' region until the end of the iteration is reached. If $Pr \ll 1$ then the Kolmogorov wavenumber, k_0, will lie in the diffusive range for the scalar field. However, as the iteration proceeds, wavenumbers will be averaged away until we will pass $k_{\phi 0}$ and again move into a region dominated by the '-5/3' power law.

Thus the theory is independent of the Prandtl number given the individual constraints on Re and Pr, as the basic assertion of the theory is that, the non-universal modes only have effective contact with the universal '-5/3' region and thus we require the fixed point of the inertial-convective range only.

The argument for the $\langle \gamma^+ \rangle_0$ terms being negligible is the same as that for the $\langle \Delta^+ \rangle_0$ case. If the ψ^+ field is approximated by a Taylor series in ϕ^+ about the upper end of the band, as in the velocity case, and if the subensemble for the conditional average is sufficiently large then the average ought to quite closely approximate that of the full ensemble. Thus, these terms will be small. By 'sufficiently large', it is intended that the turbulence scrambles any correlations existing due to stirring forces or boundary conditions. This argument breaks down at the cut between the universal and non-universal regions but this will only effect the theory if λ is small so that k is near that cut otherwise the local nature of turbulence ought to give the required decorrelation at the fixed point.

5.4 An Integral Expression for the Obukhov-Corrsin Constant

An expression for the effective diffusivity has been derived, which must be iterated under a certain recursion relation to obtain the fixed point value. Once a fixed point is found we can calculate the Obukhov-Corrsin constant, β, from the expression for the dissipation rate of second moments of the scalar field. If this is expressed for the situation after the fixed point has been reached at, say, the Nth cycle then

$$\epsilon_\phi = \int_0^{k_N} 2\chi_N(k)k^2F(k)dk \tag{5.55}$$

where $F(k)$ is the power law scalar spectrum and we input the scaled form of the fixed point of the effective diffusivity. Thus,

$$\epsilon_\phi = \int_0^{k_N} 2\alpha\epsilon^{1/3}k_n^{-4/3}\chi^*(k')k^2\beta\epsilon_\phi\epsilon^{-1/3}k^{-5/3}dk \tag{5.56}$$

and by dividing out all dimensional dependence, we obtain an equation for the Obukhov-Corrsin constant,

$$\beta\alpha^{1/2} = \left\{2\int_0^1 \chi^*(k')k'^{1/3}dk'\right\}^{-1}, \tag{5.57}$$

which depends on the Kolmogorov constant.

This result is only valid in the limit of $Re \to \infty$ since it has been assumed that the Kolmogorov spectrum is true for all wavenumbers. As in chapter 4, this can be pictured using the thought experiment of Edwards[24] where turbulence is stirred by a delta function at $k = 0$ and as $Re \to \infty$ will dissipate via a delta function at $k = \infty$. Thus the Peclet number will also be in the infinite limit provided the molecular diffusion is finite. The formula should be formally altered to

$$\beta\alpha^{1/2} = \left\{2\int_{0+}^1 \chi^*(k')k'^{1/3}dk'\right\}^{-1}, \tag{5.58}$$

5.5 Calculation and Presentation of Results

The calculation of the effective diffusivity was carried out with a modified form of the code used in the last chapter. As has already been noted, the effective viscosity forms part of the formula for the effective diffusivity and thus the two iterations must be carried out simultaneously. The only modifications needed to the outline of the code given in chapter 4 are that each time a cycle for the ω calculation is carried out, a calculation of χ must also be done using the data generated in the ω calculation.

Results for the zeroth and first orders in the Taylor series are presented. The same convergence criterion is applied to the diffusivity calculation as for the viscosity calculation. Convergence is reached when all values of both $\tilde{\omega}_{n+1}(k')$ and $\tilde{\chi}_{n+1}(k')$ are within 0.1% of the $\tilde{\omega}_n(k')$ and $\tilde{\chi}_n(k')$ values respectively. For simplicity, all results for unscaled quantities assume that $\epsilon = \epsilon_\phi = k_0 = 1$.

The results for the effective diffusivity are on the whole very similar to those for the effective viscosity. This is unsurprising when the increment formulae are compared:

$$\delta\tilde{\chi}_n(k') = \frac{1}{4\pi k'^2}\int d^3j' \frac{D_{\beta\alpha}(\mathbf{k}'-\mathbf{j}')k'_\alpha j'_\beta Q'}{\tilde{\chi}_n(hj')j'^2 + \tilde{\nu}_n(hl')l'^2} \tag{5.59}$$

and

$$\delta\tilde{\nu}_n(k') = \frac{1}{4\pi k'^2}\int d^3j' \frac{L(\mathbf{k}',\mathbf{j}')Q'}{\tilde{\nu}_n(hj')j'^2 + \tilde{\nu}_n(hl')l'^2}. \tag{5.60}$$

The underlying mathematical difference between the two problems is the linear/non-linear nature of the evolution equations in the two cases, which leads in the passive scalar transport case to a non-symmetric term being the increment term. However, the only effective difference is the angular factor which appears in the integrand since the recursion relations for the two quantities are the same. Hence, I will concentrate on the differences between the two sets of results, as many of the comments for the velocity case will apply here.

α v. λ

In figure 5.1 the values of the Kolmogorov constant, α, and the Obukhov-Corrsin constant, β, are plotted against the bandwidth parameter, λ. As for the α calculation, it can be seen that the β value has three regions. If the theory was exact one would expect it to be constant, independent of λ, since λ is a non-physical parameter. Approximations in the derivation make this impossible, though there does still exist a plateau region.

The plateau does not coincide exactly with that for the α value. Now, it must be noted that the χ^* calculation depends on ν^* and thus so does β. Hence, it is possible that the λ-independence may be spurious since we have a complex dependence on λ built in through (5.58). Thus, it will only be possible to draw conclusions about a plateau for β in the region where the two plateau sections overlap. An overlap of this type does occur for both the first and zeroth order calculations. In the first order approximation, the Obukhov-Corrsin constant is 1.02 ± 0.01 over a λ range 0.17-0.33 and this overlaps the plateau in the α graph for $\lambda : 0.25 - 0.33$. For the zeroth order calculation β is 1.22 ± 0.01 for $\lambda : 0.11 - 0.22$ and this overlaps the flat section of the α plot for $\lambda : 0.15 - 0.22$. The errors given for the constant are, as for the velocity field, a guide only since the results indicate that there is a minimum in the value.

The experimental evidence for the inertial-convective '-5/3' range is well verified (see Hill[48] for references) for mass and heat transfer. There are, however, a variety of different results for the Obukhov-Corrsin constant ranging from 0.52 to 1.93[48]. Some doubts have been cast on the experiments that yield higher values of β and thus Hill chooses 0.68 to 0.83 as the best range for a value for β.

The highest Re direct numerical simulation is by Kerr[49] which has recently been reanalysed [50] by Kerr himself and gives a value of $\beta = 1.0$ for Pr=0.5 at a Reynolds number high enough to yield about half a decade of inertial-convective behaviour. Kerr also notes a trend towards Pr independence in β which fits well

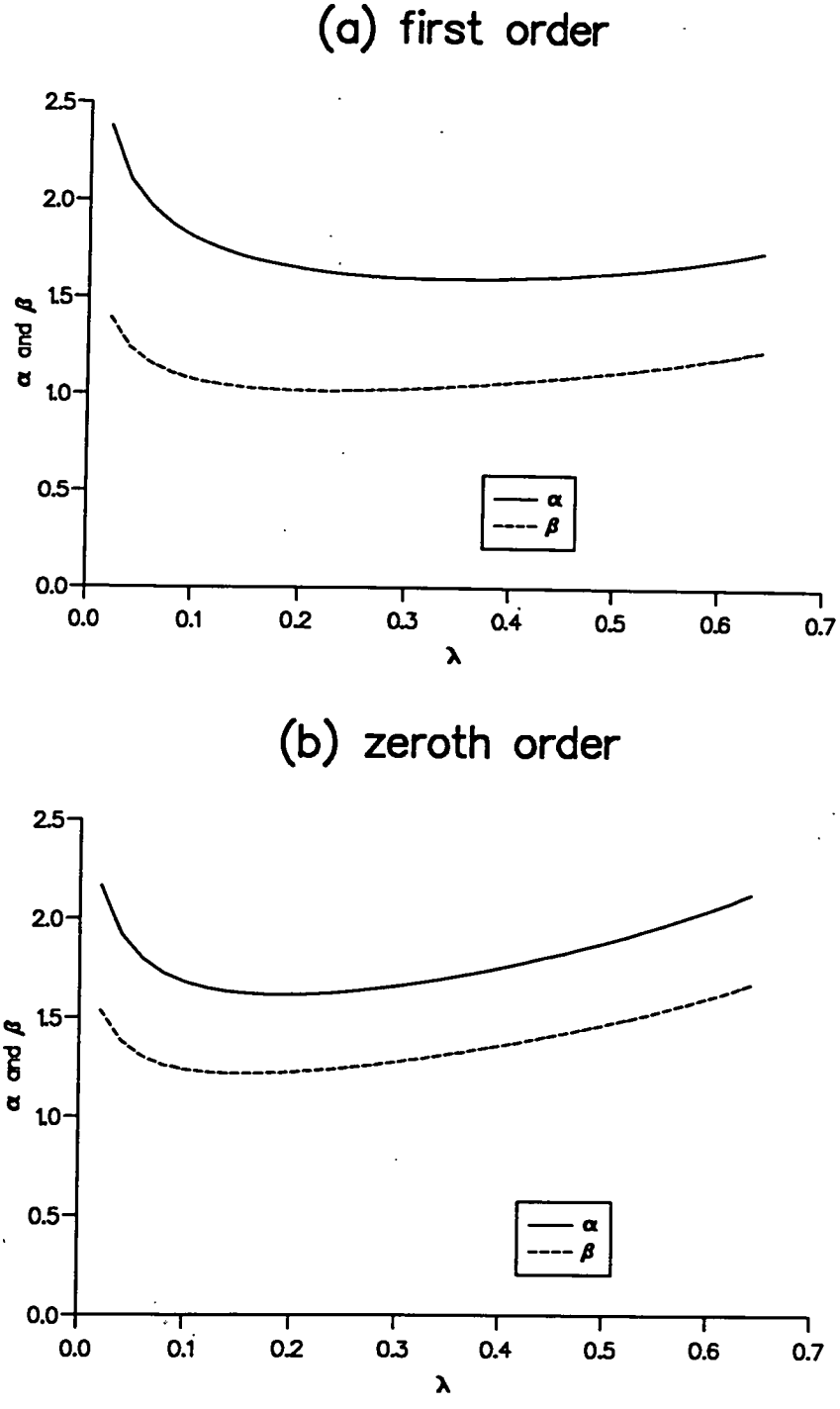

Figure 5.1: Obukhov-Corrsin and Kolmogorov constants versus bandwidth parameter

with our assumptions relating to the inertial-convective range.

Hence, it is concluded that the first order results are encouraging in their agreement with other sources and, unlike the velocity case, seem significantly better than the zeroth order results.

$\chi^*(k')$ v. k'

In figure 5.2, the results for first and zeroth order fixed point values for the effective diffusivity are plotted against scaled wavenumber. The appearance of a point of inflection at low λ and the variation of the fixed point are very similar to the results for the effective viscosity calculations. Thus, the choice of λ value to take results from will rest with the β calculation.

$\tilde{\chi}_n(k)$ v. k for various cycle nos.

Figure 5.3 shows the development of the real diffusivity culminating in the fixed point value. The roll-off at large k values develops early on. This form shows how real modes are being averaged away as the iteration continues.

$Pr_{turb}(k')$ v. k'

Figure 5.4 shows the turbulent Prandtl number against scaled wavenumber. The turbulent Prandtl number is defined

$$Pr_{turb}(k') = \frac{\nu_N(k)}{\chi_N(k)} = \frac{\nu^*(k')}{\chi^*(k')}. \tag{5.61}$$

Thus, these graphs show how the effective viscosity and diffusivity differ at the fixed point.

There are marked differences here between the zeroth and first order results. The zeroth order shows very little change with λ while the first order shows a clear trend of increasing Pr_{turb} as λ increases. Both sets of data show that, as λ decreases, the variation of the turbulent Prandtl number with k' increases. However, in the zeroth order, Pr_{turb} decreases as the upper cut is approached while in the first

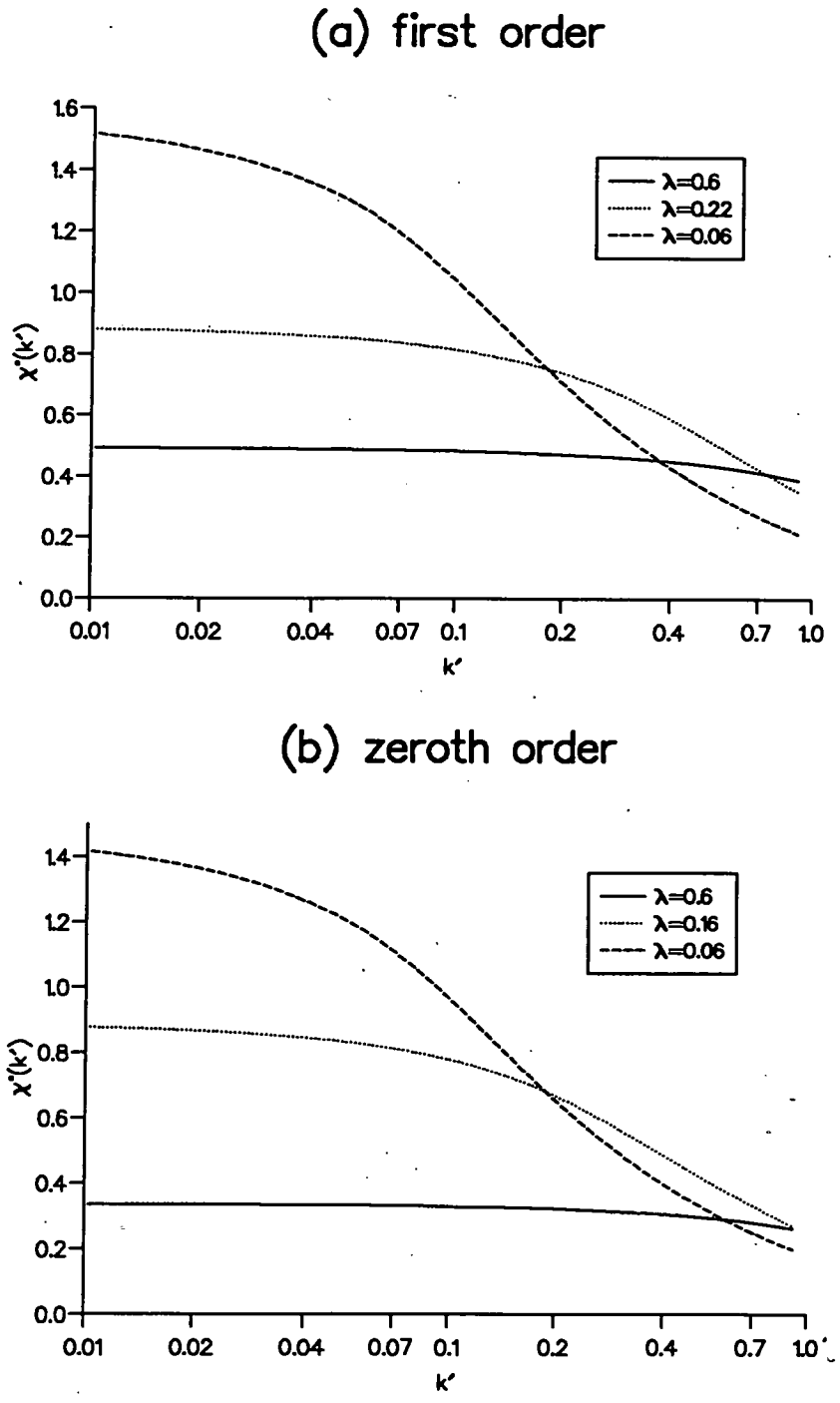

Figure 5.2: Scaled effective diffusivity versus scaled wavenumber

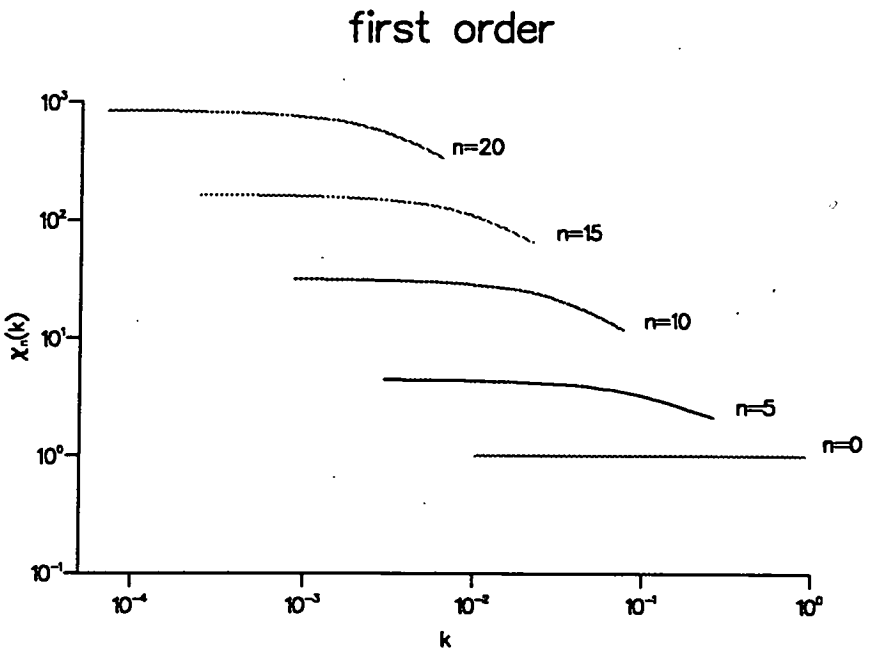

Figure 5.3: Development of the effective diffusivity under the iteration

order case Pr_{turb} increases.

Clearly, there are quantitative differences between the two cases which may explain the difference in their β predictions. The zeroth order case varies between 0.9 and 0.7 while the first order varies between 0.5 and 0.7. For the first order case of $\lambda = 0.22$ (the flat region) Pr_{turb} is quite constant at about 0.6.

The experimental values of Pr_{turb} (as might be expected by now) do not provide clear evidence of a single value for Pr_{turb}. The survey given by Hinze[51] indicates that turbulent boundary layer measurements and pipe flow measurements give constant values between 0.7 and 1 for the turbulent Prandtl number. The data that are most relevant to the present theory would be in the high local Reynolds number parts of the flow (i.e. the core region of a pipe or the wall-remote region of a boundary layer). However, the spread of results is due to the difficulty in measuring the small local spatial gradients. Thus, the result obtained here can be

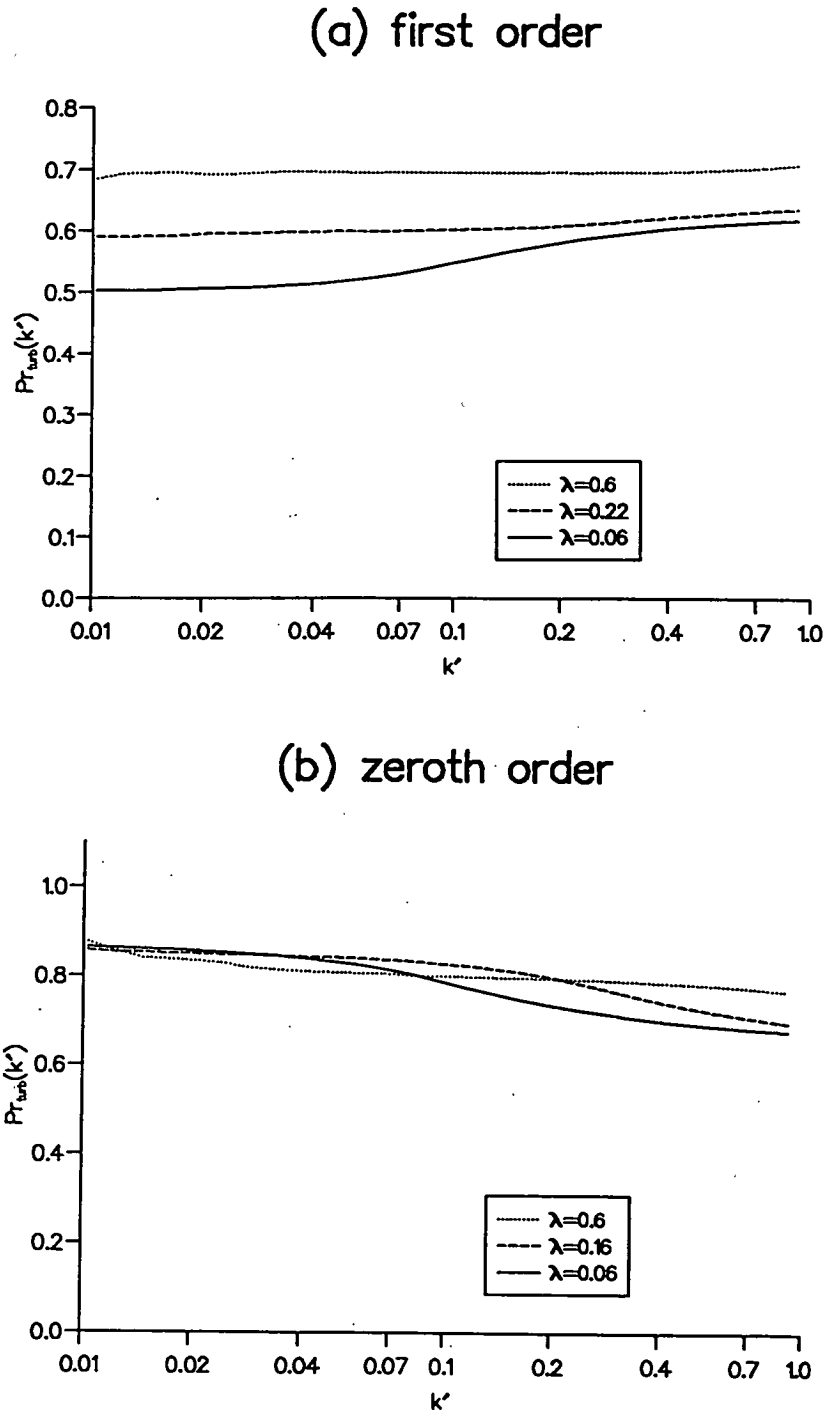

Figure 5.4: Turbulent Prandtl number versus scaled wavenumber

said to be of the correct order but not a good test of the theory.

$\chi^*(0)$ and $\chi^*(1)$ v. cycle number,n

Figure 5.5 shows the scaled diffusivity develop with the iteration at either end of the wavenumber range. For a variety of initial values of the molecular diffusivity($\widetilde{\chi}_0$=0.1, 1 and 10), the diffusivity evolves to the same value, as does the Obukhov-Corrsin constant. This shows the independence of the calculation from the non-universal aspects of the problem. The code has also been run for the same molecular Prandtl number but two different choices of $\widetilde{\nu}_0$ and $\widetilde{\chi}_0$ (i.e. Pr=1 from $\widetilde{\nu}_0 = 1$, $\widetilde{\chi}_0 = 1$ and $\widetilde{\nu}_0 = 0.1$, $\widetilde{\chi}_0 = 10.$) and these again showed independence from the initial values.

Overall, I conclude that the first order calculations give results in agreement with those expected and that the code behaves in a stable way.

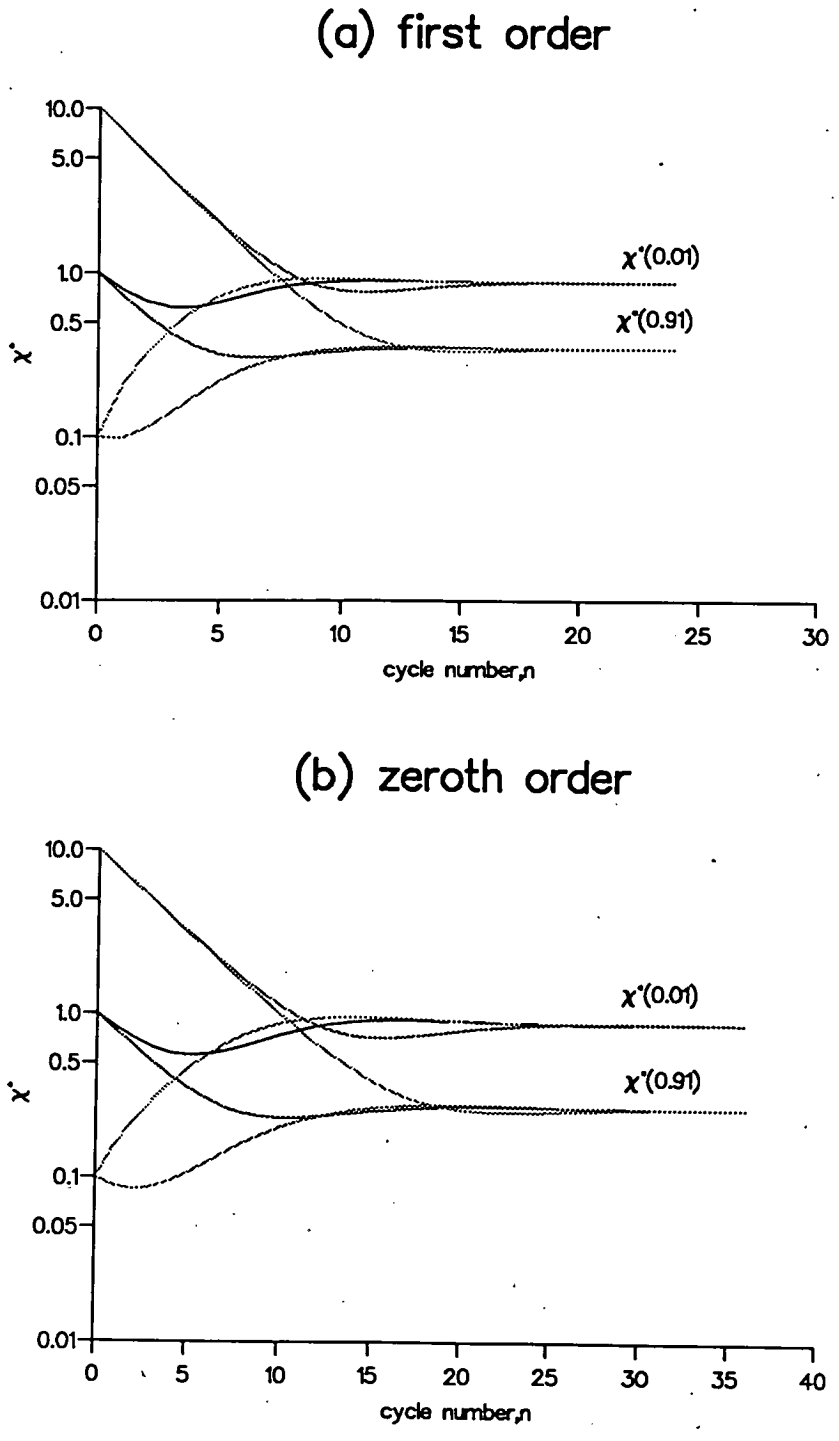

Figure 5.5: Development of the scaled effective diffusivity under the iteration

Appendix A

Summing the moment hierarchy

A.1 An upper bound ?

The series, generated in solving for the increment to viscosity, has a highly complicated general form. The convention for this discussion is that the nth term is that with n Q_v^+ terms and so 2n $\mathbf{u}^+$s. Summing this presents a formidable problem and hence, in this appendix, only an upper bound has been sought.

It has been assumed that all band variables are integrated over, consequently we start with 4n such integrals. There are n δ functions which appear with the Q_v^+ terms due to the homogeneity of the $\mathbf{v}^+$ field and these will always cancel one convolution integral. This leaves the other 2n δ functions associated with the triangularity condition to be set to one in this attempt to find an upper bound. Therefore, we must have 3n convolution integrals in total in the nth order term.

Each order in the series has many terms associated with the possible wavenumber arguments of the $\mathbf{u}^+$ to be solved for in any given conditional moment. Further, there are the permutations of the global average of $\mathbf{v}^+$ terms, when one applies the assumption that they have Gaussian statistics, breaking these up into factors

of Q_v^+. There are $(2n-1)!!$ possible ways of doing this and there are $(2n)!$ possible terms from arrangements of $\mathbf{u}^+$ arguments in the initial conditional average.

At nth order, there are 2n M terms and n D terms, the projection operators. The calculation can be further simplified by dropping this angular information in the integrals by setting all projection operators to their trace, 2. Thus, the $M_{\alpha\beta\gamma}(\mathbf{k})$ factors are bounded by $2k$.

In this way, a power series, to which summation techniques can be applied, is obtained, with the form:

$$\delta\nu_0(k)k^2 = \sum_{n=1}^{\infty}(2n)!(2n-1)!!2^{3n}kI_1^{2n-1}I_2^nI_3^{n+1} \tag{A.1}$$

where

$$I_1 = \int_{band} d^3j\,\frac{j}{\nu(j)j^2}, \tag{A.2}$$

$$I_2 = \int_{band} d^3j\,Q_v^+(j) \tag{A.3}$$

and

$$I_3 = \int_{band} d^3j. \tag{A.4}$$

It may seem on detailed consideration that I have ignored a factor of 2 generated by the choice of which variable is carried in the wavenumber argument of the $\mathbf{u}^-$ factor. I have used this to cancel the 1/2 factor arising from the isotropy argument detailed in Appendix B.

One further operation is applied that of doing each integral separately for the ML^{-1} factor and the Q_v^+ factor. This maintains the generality of this procedure to higher orders. It can be seen that otherwise as the order increases so the complexity of the terms becomes unmanageable. Of course, as each higher order is calculated so there will be another factor associated with the inversion of the linear operator (propagator) on the left-hand side of the appropriate moment evolution equation. At each higher order there will be an extra term of the form

$\nu_0 k^2$ in this propagator(see 3.2.2). I have dealt with this by replacing each ω term by its minimum in the band, ω_{min}. This creates a constant factor of $[(2n)!\omega_{min}]^{-1}$ at the nth order. This factor will cancel the (2n)! generated by the diversity of possible arrangements of $\mathbf{u}^+$ arguments in the initial conditional average.

Now, the series can be written in the form

$$\delta\nu_0(k)k^2 = C\sum_{n=0}^{\infty} a_n x^n \tag{A.5}$$

where C is the first term in the series in (A.1),

$$C = 8kI_1I_2I_3^2, \tag{A.6}$$

$$a_n = (2n+1)!! \tag{A.7}$$

and

$$x = 8I_1^2I_2I_3. \tag{A.8}$$

and finally, ω_{min} is the minimum value of $\nu_0(j)j^2$ in the band, which in the first shell will be $\nu_0 k_1^2$.

This series unfortunately is highly divergent due to the form of a_n however, it does satisfy the Poincare definition of an asymptotic expansion. The factorial nature of the divergence suggested that a Borel transformation[42] on the series might yield a guide to the upper bound.

Borel transform the series to

$$\phi(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \tag{A.9}$$

and consider

$$B(x) = \int_0^{\infty} e^{-t}\phi(xt)dt. \tag{A.10}$$

If $\phi(x)$ is convergent and $B(x)$ exists then $B(x)$ is the Borel sum of the series defined in (A.5), valid as $x \to 0^+$.

In this case,

$$\phi(x) = \sum_{n=0}^{\infty} \frac{(2n+1)!!}{n!} x^n \tag{A.11}$$

which can be summed exactly to

$$\phi(x) = (1-2x)^{3/2}. \tag{A.12}$$

Hence, for $|x| < 1/2$ this will be convergent. However,

$$B(x) = \int_0^{\infty} \frac{e^{-t}}{(1-2xt)^{3/2}} dt \tag{A.13}$$

is an improper integral which does not exist for positive x.

A.2 A sum for a model series

The Borel transformation technique is most often used for alternating series. The observation that the integral,(A.13), will not have a pole if the series alternates leads to a simple model. Instead of using the upper bound for the M factors I use $M(k) \sim (-i)2k$ which gives an alternating series of the form

$$\delta\nu_0(k)k^2 = C \sum_{n=0}^{\infty} a_n(-x)^n \tag{A.14}$$

from which the Borel integral is

$$\begin{aligned} B(x) &= C\int_0^{\infty} \frac{e^{-t}}{(1+2xt)^{3/2}} dt \\ &= C(2x)^{-3/2} e^{1/2x} \int_{1/2x}^{\infty} e^{-u} u^{-3/2} du \\ &= C(2x)^{-3/2} e^{1/2x} \Gamma(-1/2, 1/2x) \end{aligned} \tag{A.15}$$

where $\Gamma(a,z)$ is the incomplete Gamma function, which has an asymptotic expansion [43] for $z \to \infty$ of

$$\Gamma(a,z) \sim z^{a-1}e^{-z}[1 + \frac{a-1}{z} + \frac{(a-1)(a-2)}{z^2} + \ldots] \tag{A.16}$$

in $|argz| < 3\pi/2$. If the expansion is truncated at the nth order then the remainder term, $R_n(a,z) = u_{n+1}(a,z) + \ldots$, is bounded as follows

$$|R_n(a,z)| \leq |u_{n+1}(a,z)| \tag{A.17}$$

if a,z are real and $n > a - 2$. Hence, B(x) exists in the limit as $x \to 0^+$ and $B(x) \to C$.

Thus, in the limit of small x,

$$\delta\nu_0(k) \approx C. \tag{A.18}$$

Now, the factor C is equivalent in the original series to the first term as it has two M factors, one inverse propagator and is first order in a Q_v^+. Of course, x tending to zero is in fact the limit of $\lambda \to 0$ and in this limit C vanishes also. However, from the power series form above, it is clear that the difference in behaviour of terms in the series is that

$$x \sim \frac{I_1}{I_3} C \tag{A.19}$$

which implies that $x \sim C$ in this limit. Thus the first term will indeed dominate as $\lambda \to 0$.

Appendix B

Isotropy of the increment to viscosity

Here, a general property of the series for the increment term is derived which allows a simplification in writing this as a one-dimensional function. The series for the increment has a form

$$T_{\alpha\beta}(k)u_{\beta}^{-}(k). \tag{B.1}$$

Now, we can write down the general form of T as

$$\begin{aligned} T_{\alpha\beta}(k) &= A_1(k)\delta_{\alpha\beta} + A_2(k)k_\alpha k_\beta \\ &= B_1(k)D_{\alpha\beta}(k) + B_2(k)k_\alpha k_\beta, \end{aligned} \tag{B.2}$$

where

$$D_{\alpha\beta}(k) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}. \tag{B.3}$$

However, since T contains a factor

$$M_{\alpha\beta\gamma}(k) = \frac{1}{2i}[k_\beta D_{\alpha\gamma}(k) + k_\gamma D_{\alpha\beta}(k)] \tag{B.4}$$

at all orders of the series then for all k,

$$k_\alpha T_{\alpha\beta}(k) = 0. \tag{B.5}$$

Hence,

$$B_2(k) = 0. \tag{B.6}$$

Therefore, we have that

$$\begin{aligned} T_{\alpha\beta}(k)u^-_\beta(k) &= B_1(k)D_{\alpha\beta}(k)u^-_\beta(k) \\ &= B_1(k)u^-_\alpha(k), \end{aligned} \tag{B.7}$$

with

$$B_1(k) = \frac{1}{d-1} Tr T_{\alpha\beta}(k) \tag{B.8}$$

for a d dimensional system.

Appendix C

Scaled equation for the recursion relation

The aim of this appendix is to derive scaled forms for the recursion relation and the increment to viscosity. We start from the relation

$$\nu_{n+1}(k) = \nu_n(k) + \delta\nu_n(k), \tag{C.1}$$

where

$$\delta\nu_n(k) = \frac{1}{k^2}\int d^3j \frac{L(\mathbf{k},\mathbf{j})\{Q(l)|_{l=k_n} + (l-k_n)\frac{\partial Q(l)}{\partial l}|_{l=k_n}\}}{\nu_n(j)j^2 + \nu_n(|\mathbf{k}-\mathbf{j}|)|\mathbf{k}-\mathbf{j}|^2}, \tag{C.2}$$

restricted to $0 \leq k \leq k_{n+1}$.

Now, from (3.53),

$$\nu_{n+1}(k) = \alpha^{1/2}\epsilon^{1/3}k_{n+1}^{-4/3}\tilde{\nu}_{n+1}(k'), \tag{C.3}$$

which defines

$$k' = k_{n+1}^{-1}k. \tag{C.4}$$

So with this notation,

$$\begin{aligned}\nu_n(k) &= \alpha^{1/2}\epsilon^{1/3}k_n^{-4/3}\tilde{\nu}_n(k/k_n) \\ &= \alpha^{1/2}\epsilon^{1/3}k_n^{-4/3}\tilde{\nu}_n(hk').\end{aligned} \tag{C.5}$$

Scaling the increment, (C.2),

$$\delta\nu_n(k) = \frac{1}{(k_{n+1}k')^2}\int d^3j' \frac{k_{n+1}^3}{4\pi} \frac{(k_{n+1}^2 L(\mathbf{k}',\mathbf{j}'))\alpha\epsilon^{2/3}k_{n+1}^{-11/3}Q'}{\alpha^{1/2}\epsilon^{1/3}k_n^{-4/3}(\tilde{\nu}_n(hj')(k_{n+1}^2 j'^2) + \tilde{\nu}_n(hl')(k_{n+1}^2 l'^2))}, \tag{C.6}$$

where $j = k_{n+1}j'$ and $l = |\mathbf{k}-\mathbf{j}| = k_{n+1}l'$. This defines the scaled increment as

$$\delta\nu_n(k_{n+1}k') = \alpha^{1/2}\epsilon^{1/3}k_n^{-4/3}h^{-8/3}\delta\tilde{\nu}_n(k'), \tag{C.7}$$

where

$$\delta\tilde{\nu}_n(k') = \frac{1}{4\pi k'^2}\int d^3j' \frac{L(\mathbf{k}',\mathbf{j}')Q'}{\tilde{\nu}_n(hj')j'^2 + \tilde{\nu}_n(hl')l'^2} \tag{C.8}$$

for the wavenumber bands

$$0 \le k' \le 1; 1 \le j', l' \le h^{-1}.$$

Now, (C.1) becomes

$$\begin{aligned}\alpha^{1/2}\epsilon^{1/3}k_{n+1}^{-4/3}\tilde{\nu}_{n+1}(k') &= \alpha^{1/2}\epsilon^{1/3}k_n^{-4/3}\tilde{\nu}_n(hk') \\ &\quad + \alpha^{1/2}\epsilon^{1/3}k_n^{-4/3}h^{-8/3}\delta\tilde{\nu}_n(k'),\end{aligned} \tag{C.9}$$

and since $k_{n+1} = hk_n$,

$$h^{-4/3}\tilde{\nu}_{n+1}(k') = \tilde{\nu}_n(hk') + h^{-8/3}\delta\tilde{\nu}_n(k'). \tag{C.10}$$

Thus, the scaled recursion relation can be written,

$$\tilde{\nu}_{n+1}(k') = h^{4/3}\tilde{\nu}_n(hk') + h^{-4/3}\delta\tilde{\nu}_n(k'). \tag{C.11}$$

Note that this is different from the earlier forms used in Iterative Averaging[29].

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